Statistical Methods: from Physics to Finance

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TRIUMF, 2011

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The PIENU Experiment

Precise Measurement of the Branching Ratio:

\[ B = \frac{\pi \rightarrow e \nu}{\pi \rightarrow \mu \nu \nu \rightarrow e} \]

- TRIUMF Based: pion beam thanks to the 500 MeV Cyclotron
- Precision Experiment
- Uncover Physics beyond the Standard Model
The intersections between Physics and Finance/Economy are many!
A very personal (and limited) selection of topics:

I) Introduction and Short History
II) Some Physics Problems and Probability Distributions
   - Heavy Tails: Multiple Scattering
   - Gaussians: Heat Flow and Diffusion
   - Brownian Motion

III) Stochastic Calculus
IV) Stable Distributions
V) Stocks
VI) Derivatives: Options
VII) Summary & Conclusions
Is there any logic behind Economics or the Stock Market?
Is there any logic behind Economics or the Stock Market?

Just a normal day at the nation's most important financial institution...

I've got a stock here that could really excel.

Really excel?

Sell?

Sell?

Sell!

Sell!

Sell!

Sell!

This is madness! I can't take anymore.

Goodbye.
Is there any logic behind Economics or the Stock Market?
Historical Overview

I) R. Brown (1773-1858): Observation of random motion in pollen samples
II) L. Bachelier (1870-1946): First attempt to model stocks movements
III) A. Einstein (1905): First model of the brownian motion
IV) Mandelbrot and Pareto Distributions
V) K. Ito and R. Stratonovich (~1950-60): Calculus with random variables
VI) Black, Merton, Scholes: Stochastic model for options (~1973)

1997 Nobel Prize in Economics

Now:
“Econophysics”: tries to apply physics methods to Finance and Economy:

- Stochastic Processes
- Statistical Physics
- Agent-based Models
- Statistical Analysis
- Feynman’s Path Integrals
PART 1:

PHYSICS MODELS
and
PROBABILITY DISTRIBUTIONS:
EXPONENTIALS vs POWER LAWS
Exponentials and Power-Laws Distributions

Probability Distribution: \( p(x) \)

Probability: \( P(a < x < b) = \int_a^b p(x) \, dx \)

Moments: \( \mu_k = \int x^k \, p(x) \, dx \)
Exponentials and Power-Laws Distributions

Probability Distribution: $p(x)$

Probability: $P(a < x < b) = \int_a^b p(x) \, dx$

Moments: $\mu_k = \int x^k \, p(x) \, dx$

"Tails"

Exponential (e.g. Gaussian)

$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

- All the moments are finite
- Central Limit Theorem
- Limit of many distributions
Exponentials and Power-Laws Distributions

Probability Distribution: \( p(x) \)

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\]

- All the moments are finite
- Central Limit Theorem
- Limit of many distributions

Power-Law (e.g. Lorentzian)

\[
p(x) = \frac{1}{\pi} \frac{\gamma}{(x-\mu)^2 + \gamma^2}
\]

- Undefined moments \( k > 0 \)
- CLT cannot hold
Multiple Scattering

Used e.g. in:
- Detector Physics and Simulation
- Medical Physics

Two physics mechanisms:
- EM Force
- Strong Force

\[ \theta = \frac{13.6 \text{MeV}}{\beta c p} z \sqrt{x/X_0} \left[ 1 + 0.038 \ln \left( \frac{x}{X_0} \right) \right] \]
**Heat Transfer and Diffusion**

**Fourier's Law**

\[ q = -k \nabla u \]

**Thermodynamics:**

\[ Q = c_p \rho u \]

If a change in internal energy in a material is given only by heat flux across the boundaries in a space/time region: \( x - \delta x \leq \xi \leq x + \delta x \); \( t - \delta t \leq \tau \leq t + \delta t \)

\[
k \int_{t-\delta t}^{t+\delta t} \left[ \frac{\partial u}{\partial x} (x + \delta x, \tau) - \frac{\partial u}{\partial x} (x - \delta x, \tau) \right] d \tau = k \int_{x-\delta x}^{x+\delta x} \int_{t-\delta t}^{t+\delta t} \frac{\partial^2 u}{\partial \xi^2} d \xi d \tau
\]

\[
c_p \rho \int_{x-\delta x}^{x+\delta x} \left[ u(\xi, t + \delta t) - u(\xi, t - \delta t) \right] d \xi = c_p \rho \int_{x-\delta x}^{x+\delta x} \int_{t-\delta t}^{t+\delta t} \frac{\partial u}{\partial \tau} d \xi d \tau
\]

By energy conservation:

\[
\frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} = 0
\]
Solution of the Heat Equation

\[
\frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} = 0 \quad \rightarrow \quad \frac{\partial u}{\partial t}(x,t) - k \nabla^2 u(x,t) = 0
\]

Temperature

\[
\frac{\partial c}{\partial t}(x,t) - k \nabla^2 c(x,t) = 0 \quad \text{Concentration}
\]

Green's Function (Propagator):

\[
G(x,t) = \frac{1}{\sqrt{4\pi k t}} e^{-\frac{x^2}{4kt}}
\]

General Solution:

\[
u(x,t) = \int G(x-y,t) f(y,t=0) \, dy
\]
Why Physicists are interested in power-law distributions?

Random events usually give rise to exponential tails: understood. If there are power law tails, something interesting is happening:

1) Critical State?
During phase transitions, correlation functions have power-law behavior. This means e.g. that fluctuations are happening at all scales (see 2)).

2) Scale-free system
Power-law distributions are scale free: 

\[ P(sx) = a(s) P(x) \]

Only one candidate distribution: 

\[ P(x) = C x^{-\tau} \]

Partial proof:

\[ P(sx) = C(sx)^{-\tau} = C s^{-\tau} x^{-\tau} = a(s) P(x) \]
Brownian Motion

R. Brown (1773-1858): Observation of random motion in pollen samples

A. Einstein (1905): First quantitative explanation of the brownian motion

\[ \frac{\partial u}{\partial t} - D \frac{\partial^2 u}{\partial x^2} = 0 \]

\[ D = \frac{RT}{N} \frac{1}{6 \pi k r} \]  
(For spherical particles)

\[ \sqrt{\langle x^2 \rangle} = \sqrt{2} D t \]

AE estimate: \( \approx 6 \mu m / \) minute @ 17°C, \( N=10^{23} \)

\[ N = \frac{1}{\sqrt{\langle x^2 \rangle}} \frac{RT}{3 \pi k P} \]

Opportunity to estimate \( N \)!

“... It is hoped that some enquirer may succeed shortly in solving the problem suggested here, which is so important in connection with the theory of heat.”

Berne, May 1905 (Received, 11 May 1905).
A Simple Model for the Brownian Motion: Symmetric Random Steps

\[ \langle \Delta x \rangle = 0 \]

\[ \text{Var} (\Delta x) = \langle (\Delta x)^2 \rangle - (\langle \Delta x \rangle)^2 = \frac{1}{2} h^2 + \frac{1}{2} (-h)^2 = h^2 \]

\[ t = N \Delta t \Rightarrow Nh^2 = \frac{th^2}{\Delta t} \]

If we want the variance at \( t \) being:

\[ \sigma^2 t \Rightarrow h = \sigma \sqrt{\Delta t} \]

\[ p( (N + 1) \Delta t, ih) = \frac{1}{2} p( N \Delta t, (i - 1) h) + \frac{1}{2} p( N \Delta t, (i + 1) h) \]

\[ \partial_t p(t, x) = \frac{\sigma^2}{2} \partial_x^2 p(t, x) \]

\[ p(t, x) = \frac{1}{\sqrt{2\pi(t-s)}} e^{\frac{-(x-x_0)^2}{2\sigma^2(t-s)}} \]
Physics: Summary

I) Random processes often encountered in physics (Probability/Distributions).

II) Connection to the diffusion (or heat) equation.

III) Not all random processes have gaussian tails!

IV) “Fat” tails are not always easy to explain: associated with “complex” systems, phase transitions, critical states, scale invariance, …
   In general: something interesting for a physicist!

V) More formal methods needed to treat stochastic processes.
PART 2:

STOCHASTIC PROCESSES
Brownian Processes

A standard brownian process $B_{t}$, $t > 0$ is a continuous stochastic process in $t$, with the following properties:

I) $B_{0} = 0$ with probability 1

II) The increments are independent, i.e.: if $0 < t_{0} < \ldots < t_{k}$, then $B_{t_{1}} - B_{t_{0}}, \ldots, B_{t_{k}} - B_{t_{k-1}}$ are independent.

III) For every $s, t$, with $0 < s < t$, the increments $B_{t} - B_{s}$ are distributed like $N(0, t - s)$

In a non-standard process, $N(0, \sigma^{2}(t - s))$
It is not possible to define derivatives in a stochastic process (jumps!)
It is possible to define the integral: the Itô (Stratonovich) integral:

General Stochastic Process:

\[ X_t(\omega) = X_{t_0} + \int_{t_0}^t F(\tau, \omega) \, dt + \int_{t_0}^t G(\tau, \omega) \, dW_\tau(\omega) \]

Stochastic Differential:

\[ dX_t = F(t, \omega) \, dt + G(t, \omega) \, dW_t \]
The Ito/Stratonovich Integral

Riemann-Stieltjes Integral:

\[
\int_0^t f(t)dg(t) = \lim_{n \to \infty} \sum_{i=1}^n f(\tau_i)(g(t_{i+1}) - g(t_i))
\]

If \( g \) is smooth, the limit is independent on the choice of \( \tau \).
It is not true for stochastic processes!

Ito Integral: \( \tau_i = t_i \)

Example: \( \int_0^t W_t dW_t = \frac{1}{2} W_t^2 - \frac{1}{2} (W_0 + t) \)

Stratonovich Integral: \( \tau_i = \frac{1}{2} (t_{i+1} + t_i) \)

Example: \( \int_0^t W_t dW_t = \frac{1}{2} W_t^2 - \frac{1}{2} W_0 \)
The Ito's Formula

**Stochastic Process:**

\[ dX_t = F(t, .) dt + G(t, .) dW_t \]

**...a function of it:**

\[ Y_t = U(t, X_t) \]

**The Differential:**

\[ dY(t) = \left\{ \partial_t U(t, X_t) + F(t) \partial_X U(t, X_t) + \frac{1}{2} G^2(t) \partial^2_X U(t, X_t) \right\} dt + G(t) \partial_X U(t, X_t) dW_t \]

Ito's formula tells us how to calculate derivatives of composite functions. The result is different wrt the classical calculus.
The Ito's Formula

Stochastic Process:

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Ito's formula tells us how to calculate derivatives of composite functions. The result is different wrt the classical calculus.
Stochastic Processes

\[ dX_t = F(t, X_t) \, dt + G(t, X_t) \, dW_t \]

\[ dX_t = \sigma^2 \, dW_t \]  \hspace{1cm} \text{Simple Brownian Motion (L. Bachelier Model)}

\[ dX_t = \mu \, dt + \sigma^2 \, dW_t \]  \hspace{1cm} \text{Brownian Motion with Drift}

\[ dX_t = \mu X_t \, dt + \sigma^2 X_t \, dW_t \]  \hspace{1cm} \text{Geometric Process}

\[ dX_t = \lambda (\mu - X_t) \, dt + \sigma^2 X_t \, dW_t \]  \hspace{1cm} \text{“Return to the mean” process}

\[ dX_t = -\lambda X_t \, dt + \sigma^2 \, dW_t \]  \hspace{1cm} \text{Ornstein-Uhlenbeck (Fluctuation around zero)}
Summary

I) There is a powerful formalism for treating stochastic processes

II) Used in many scientific areas:
- Condensed Matter Physics
- Meteorology/Oceanography
- Chemistry
- Connections with Quantum Mechanics / QFT
- Turbulence / Hydrodynamics
- ....

III) Financial time series look like random walks: is it possible to apply these ideas to finance or economy?
PART 3:

STOCK MARKET
Financial Time Series

I) Can we describe financial time series with stochastic processes?

II) If yes, what are the properties of such a process?

III) In the market, not only stocks are exchanged: derivates. Models?

IV) Not only Finance: Economic Systems

Any relation with physics models?
THE VARIATION OF CERTAIN SPECULATIVE PRICES*

BENOIT MANDELBROT†

I. INTRODUCTION

The name of Louis Bachelier is often mentioned in books on diffusion process. Until very recently, however, few people realized that his early (1900) and path-breaking contribution was the construction of a random-walk model for security and commodity markets.1 Bachelier’s simplest and most important model goes as follows: let \( Z(t) \) be the price of a stock, or of a unit of a commodity, at the end of time period \( t \). Then it is assumed that successive differences of the form \( Z(t + T) - Z(t) \) are independent, Gaussian or normally distributed, random variables with zero mean and variance proportional to the differencing interval \( T \).2

Despite the fundamental importance of Bachelier’s process, which has come to be called “Brownian motion,” it is now obvious that it does not account for the abundant data accumulated since 1900 by empirical economists, simply because the empirical distributions of price changes are usually too “peaked” to be relative to samples from Gaussian populations.3 That

2 The simple Bachelier model implicitly assumes that the variance of the differences \( Z(t + T) - Z(t) \)
**Observables**

Nowadays: huge database of (high frequency) financial data. Opportunity to study (“data analysis”) the market and try to “understand” it. What are the important quantities to study? No reference units!

- **Price** \[ Y(t) \]
- **Price Change** \[ Z(t) = Y(t + \Delta t) - Y(t) \]
- **Return** \[ R(t) = \frac{Y(t + \Delta t) - Y(t)}{Y(t)} \]
- **Log-Change** \[ S(t) = \ln Y(t + \Delta t) - \ln Y(t) \]
Distribution of the increments (returns)

- Not Gaussian! It is not a Brownian Motion.

- Leptokurtic: heavy tails  \( K = \frac{\mu_4}{\sigma^4} - 3 > 0 \)  
  \[ \mu_i = \int (x - x_0)^i p(x) dx \]

- Almost no skewness \( S = \mu_3 \approx 0 \)
Correlation Functions

N-point Functions:
\[ \langle x(t) \rangle = \int dx x f(x, t) \]
\[ \langle x(t_1) x(t_2) \rangle = \int dx_1 dx_2 x_1 x_2 f(x_1, t_1, x_2, t_2) \]
\[ \langle x(t_1) x(t_2) ... x(t_n) \rangle \]

Stationarity:
\[ P[x(t)] = P[x(t+\tau)] \]

Autocorrelation:
\[ C(t_1, t_2) = \langle x(t_1) x(t_2) \rangle - \mu(t_1) \mu(t_2) = R(\tau) - \mu^2 \quad \tau = t_2 - t_1 \]

Memory:
\[ M = \int_0^\infty R(\tau) d\tau \]
\[ \text{Finite} \]
\[ \text{Infinite} \]
\[ \text{Undetermined} \]

Power Spectrum:
\[ S(f) = \int_{-\infty}^{\infty} R(\tau) e^{-2i\pi f \tau} d\tau = \frac{2\sigma^2 \tau_c}{1 + (2\pi ft_c)^2} \]
\[ S(f) \approx \frac{1}{f^2} \quad \text{Brownian} \]
\[ S(f) \approx \frac{1}{f^N} \quad 0 < N < 2 \]
Correlation Functions (Single Stock)

Autocorrelation Function (Coca-Cola Company):
Very fast decay (<1 day).
Arbitrage difficult.
No memory

Spectral Density: (Coca-Cola Company):
\[ S(f) \approx \frac{1}{f^2} \]
Compatible with a random walk

From: Stanley, Mantegna
Correlation Functions (Market Index)

Autocorrelation Function (S&P 500):
Very fast decay (Corr time <4 min.).
Arbitrage difficult.
No memory

Spectral Density: (S&P 500):
\[ S(f) \approx \frac{1}{f^2} \]
Compatible with a random walk

From: Stanley, Mantegna
Higher Order Correlations

- Pairwise independence does not imply independent random variables!
- Look for high-order functions of the price change: e.g. the VOLATILITY.

LONG RANGE CORRELATION!
Power Law with exponent \(~0.3\)
Volatility: Summary

I) No “average” volatility exists
II) Volatility is difficult to estimate (in the future)
III) Clustering (Intermittency?).
IV) Memory.
V) ARCH / GARCH Models
Levy Stable Distributions

**Definition:** $x$ is a stable random variable if for every positive $A,B$, exist $C,D$ so that:

$$(Ax_1 + Bx_2) \approx Cx + D$$

Where $x_1$ and $x_2$ are copies of $x$.

**Theorem:** The sum of independent, identically distributed random variables converge to a stable distribution as the number of iid variables tends to infinity.

**Stable (Levy) Distributions** $S_\alpha(\gamma, \beta, \mu)$

$$\ln \varphi(k) = \begin{cases} 
i \mu k - \gamma |k|^{\alpha} \left[1 - i \beta \frac{k}{|k|} \tan(\pi / 2 \alpha) \right] & \alpha \neq 1 \\
i \mu k - \gamma |k| \left[1 + i \beta \frac{k}{|k|} (2/\pi) \tan (|k|) \right] & \alpha = 1 \end{cases}$$

$0 < \alpha < 2 \quad \gamma > 0 \quad \mu \in \mathbb{R} \quad \beta \in (-1, +1)$
Power Law (Fat) Tails

Some known Analytic Forms:

- \( \alpha = 2 \) \( \Rightarrow \) Gaussian
- \( \alpha = 1, \beta = 1 \) \( \Rightarrow \) Landau
- \( \alpha = \frac{1}{2}, \beta = 1 \) \( \Rightarrow \) Levy
- \( \alpha = 1, \beta = 0 \) \( \Rightarrow \) Lorentz

Stable Distributions have fat tails:

\[
\lim_{x \to \infty} P(X > x) \approx \frac{\Gamma(1+\alpha) \sin(\pi \alpha/2)}{\pi |x|^{1+\alpha}} \approx \frac{1}{|x|^{1+\alpha}}
\]
The “stylized” Facts

I) Fat Tails
Large returns follow a fat-tailed distribution $P(r) \sim r^{-\alpha}$. Generally, $\alpha > 2$, therefore the variance is finite. This questions the use of Levy stable distributions. Truncated Levy pdfs have been proposed, among the many other proposals. The issue is still debated.

II) Asymptotic Normality
When the time scale of the returns is increased, the distribution tends to the normal one. Conclusion: the shape of the distribution changes with the time scale and therefore the underlying stochastic process has a non trivial temporal structure.

III) No linear autocorrelations (efficient market, no arbitrage)
The autocorrelation falls quickly to 0: after ~15min it is practically =0. There is no memory effect: support of the “efficient market hypothesis”. Over longer timescales, (months) there are some (non conclusive) hints of autocorrelation.

IV) Clustered Volatility
It is evident that the volatility of the market is not constant in time and there is autocorrelation: high volatility periods cluster in time and large changes are followed by large changes.
Stochastic Processes in Finance: Conclusions

Data:
- Today, thanks to computers, we have a very large dataset of financial data to study.
- It was not possible at the times of Bachelier!

Stochastic Process:
- It is not a simple Brownian Motion (Bachelier Model)
- Exhibits heavy (fat) tails
- Very short correlation time (No memory)
- Volatility has longer correlation (Memory?)
- ...

Models:
- Brownian Motion (Not accurate, possibility of negative prices)
- Lognormal Process (Better, but not fully accurate)
- Ornstein-Uhlenbeck Process
- Levy, Truncated Levy, Tsallis, ... (Better heavy tails, but debated)
- ARCH / GARCH , .... (Powerful, simple: worthed the 2003 Nobel Price!)
The Black-Merton-Scholes Model

Robert C. Merton

Myron S. Scholes

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1997 was awarded jointly to Robert C. Merton and Myron S. Scholes "for a new method to determine the value of derivatives"


Black, F., 1989, "How We came Up with the Option Formula", The Journal of Portfolio Management, Vol. 15, pp. 4-8


Derivates: The Options

Definition:
In finance, an option is a derivate financial instrument that establishes a contract between two parties concerning the buying or selling of an asset at a reference price.

“Put” (“Call”) European Options:
The owner of the option has the right (but no obligation) to sell (buy) a certain good at a certain time for a certain price.

Other jargon terms:
Underlying (Asset): A stock for a simple option (can be any good).

Strike Price: Price for buying or selling the underlying.

Expiration Date: When the option might be used.

American Options: The option can be used at any time before expiration.

Exotic Options: All other kind of options (e.g. asian options, ...).
The Black-Merton-Scholes Option Pricing Model

Hypoteses:

I) The short-term interest rate $r$ is known and constant and it is possible to ask for money at that rate.

II) The strike price $X$ is known and constant

III) The stock price $S$ follows a geometric stochastic process

$$dS_t = \mu S_t dt + \sigma^2 S_t dW_t$$

Where $W$ is a Wiener process, $\mu$ is the expected return, $\sigma$ the volatility.

IV) The stock does not pay dividends

V) No transaction costs and no limits to the short-selling

VI) There is no arbitrage
The Black-Merton-Scholes Option Pricing Model

Build a portfolio made of stocks and options:

\[ V = S - \frac{1}{\Delta} c \]

Consider its variation:

\[ dV = dS - \frac{1}{\Delta} dc \]

Apply Ito's Lemma to \( c(t,S) \):

\[ dc = \left[ \frac{\partial c}{\partial t} + \frac{1}{2} \left( \frac{\partial^2 c}{\partial S^2} \right) \sigma^2 S^2 \right] dt + \frac{\partial c}{\partial S} \sigma S dW \]

Substituting:

\[ dV = \frac{-1}{\Delta} \left[ \frac{\partial c}{\partial t} + \frac{1}{2} \left( \frac{\partial^2 c}{\partial S^2} \right) \sigma^2 S^2 \right] dt \]

Equivalence to a risk-free portfolio: (no arbitrage)

\[ \frac{-1}{\Delta} \left[ \frac{\partial c}{\partial t} + \frac{1}{2} \left( \frac{\partial^2 c}{\partial S^2} \right) \sigma^2 S^2 \right] dt = \left( S - \frac{1}{\Delta} c \right) rdt \]

After some algebra:

\[ \frac{\partial c}{\partial t} = rc - rS \frac{\partial c}{\partial S} - \frac{1}{2} \left( \frac{\partial^2 c}{\partial S^2} \right) \sigma^2 S^2 \]
Solution of the BSM Equation

The solution of the BSM equation is:

\[ c(S, t) = SN(d_2) - X e^{-r(T-t)} N(d_2) \]

\[ \ln(S/X) + (r + \frac{1}{2} \sigma^2)(T-t) \]

\[ d_1 = \frac{\ln(S/X) + (r + \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}} \]

\[ d_2 = d_1 - \sigma \sqrt{T-t} \]

\[ N(d_{i=1,2}) = \int_{-\infty}^{d_i} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \]

Boundary condition at \( t=T \):

\( c = \max(S - K, 0) \)

(for a call option)
Physics Interpretation of the BMS Equation

\[ \partial_t c + \frac{1}{2} \sigma^2 S^2 \partial_{SS} c + rS \partial_S c - rc = 0 \]

Diffusion Term

Convection Term

Reaction Term

Example: Dispersion of a pollutant in a river:

- Dispersion: Diffusion Term
- Water Flow: Convection Term
- Absorption (e.g. by sand): Reaction Term
The BMS equation IS the heat equation!

\[ c(S,t) \text{ is solution of the BMS equation } \partial_t c + \frac{1}{2} \sigma^2 S^2 \partial_{SS} c + rS \partial_S c - rc = 0 \]

\[ u(x,\tau) \text{ is solution of the heat equation } \partial_\tau u = \partial_{xx} u \]

Substitution:

\[ S = e^x \]
\[ \tau = \frac{1}{2} \sigma^2 (T - t) \]
\[ k = \frac{2r}{\sigma^2} \]

\[ c(S,t) = \exp \left[ -\frac{1}{2} (k - 1)x - \frac{1}{2} (k + 1)^2 \tau \right] u(x,\tau) \]
BMS and Montecarlo (N=10): The “Greeks”

\[ c = c(S) \]

\[ \Delta = \frac{\partial c}{\partial S} \]

\[ V = \frac{\partial c}{\partial \sigma} \]

\[ \Theta = \frac{\partial c}{\partial T} \]
BMS and Montecarlo (N=100): The “Greeks”

\[ c = c(S) \]

\[ \Delta = \frac{\partial c}{\partial S} \]

\[ V = \frac{\partial c}{\partial \sigma} \]

\[ \Theta = \frac{\partial c}{\partial T} \]
BMS and Montecarlo (N=1000): The “Greeks”

\[ c = c(S) \]

\[ \Delta = \frac{\partial c}{\partial S} \]

\[ V = \frac{\partial c}{\partial \sigma} \]

\[ \Theta = \frac{\partial c}{\partial T} \]
Summary

I) Nowadays, statistical analysis of financial time series possible:
   - Probability Distributions (Returns, Rates, ...) : Heavy-tailed distributions
   - Time correlations, cross-correlations.
   - Spectral densities
   - ...

II) Modeling of the time series with stochastic models:
   - Brownian motion (Log-normal process)
   - (Truncated) Levy processes
   - Mixed processes (jump-diffusion,...)

III) Derivates:
   - BMS Model and all the possible generalizations, Delta-Hedging
   - PDE methods (also numerical)
   - Montecarlo methods
   - Feynman Path Integrals !!
Conclusion

Do really a physics theory of finance (economics) exists?

(My) answer is: NO:

Often there are only some analogies and the economic system do not resemble closely any physics system (yet). There is no “econophysics” theory, like quantum mechanics or general relativity: it is more like a collection of methods, many of them borrowed from or used by physicists.

BUT:
Physicists have skills often very useful if applied in this field:

I) Strong mathematical background
II) Strong computational background
III) Construction of models
IV) Statistical analysis of (huge) datasets
V) ....
Ettore Majorana (1906 – 1938 – ?)

Il valore delle Leggi Statistiche nella Fisica e nelle Scienze Sociali
(Unpublished)

*The Value of the Statistical Laws in Physics and in the Social Sciences*

“The study of the relationships, true or supposed, between physics and the other sciences, had always been of big interest because of the special influence physics had in modern times on the scientific thinking.”

“It is therefore important that the principles of quantum mechanics brought us to recognize the statistical character of the ultimate laws of the elementary processes.”

*My “free” translation!*
Grazie!
Thank You!
Merci!