PHYSICS IN ACTION

Lesson Workbook by Philip Freeman and Marcello Pavan

Electromagnetism and Circular Motion in a Cyclotron

What is a Cyclotron and Why Build One?

The TRIUMF (the TRI University Meson Facility) cyclotron is one of a large class of particle accelerators. People used to call these "atom smashers" and that's not a bad name (though these days a lot more than that is done with accelerators). One of the chief uses of the TRIUMF cyclotron is indeed to smash atoms, but not just as random acts of vandalism!

What TRIUMF does is to take particles (protons) and accelerate them to EXTREMELY high speeds (high energies). These speeds are needed, because the higher the momentum of a particle the smaller its 'size' is, and only at very high energies is a proton 'small' enough to investigate the inside of atoms. Further at very high energies protons can be used to create mesons, particles beyond the "proton, electron, neutron" description of the atom and which we know play an important role in holding the nucleus together.

There are other very useful things that we can do with these high energy protons, and TRIUMF is doing important work in medicine both using the high energy protons to destroy cancers and by creating special isotopes which can be used to track the processes in living things.

How a Cyclotron Works

To produce the very high-energy protons that we want out of our cyclotron we need to find a way to:

1) Get protons	Lesson 1 – 3	ION SOURCE AND INJECTION
2) Make them go very fast	Lesson 4	CYCLOTRON
3) Put them where we want them	Lesson 5	EXTRACTION AND BEAMLINES
4) Use them	Lesson 6	BEAM PRODUCTION & FILTERING

This video addresses each of these four areas in the lessons shown in the above table.

Here, we will look at each of these processes in the next sections, but for now let's try a simple analogy to get the "big picture".
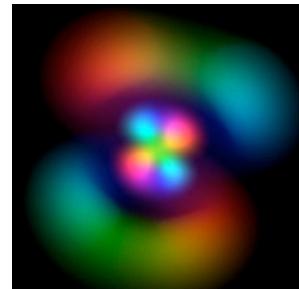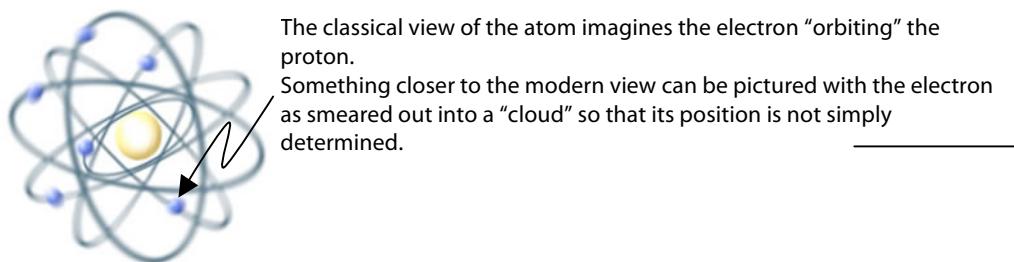
An Analogy: Before the invention of guns and the like one type of weapon used in hunting or war was the sling. It was a strip of cloth in which a stone or lead weight was placed. The stone was then swung around until going at very high velocity, and finally released to hit a target. This is almost exactly like a cyclotron!

1) **Get a stone:** If you are using a sling you need not just any stone, but one of the right shape and size. Even though all protons are identical we still need to make and select particular "stones" for our cyclotron because to control and accelerate the particles we need them to be charged, so we create ions. The cyclotron uses H- ions, which we make, give them an initial 'push' and then steer into the cyclotron.

2) **Make it go very fast:** A sling holds the stone it contains so that it is forced to go into a circular path. Each time around the person pulls forward a little on the stone, making it go faster each time until it has a high speed. In the cyclotron a magnetic field is used to hold the ions in a circular path, and each time around an electric field is used to pull it forward a little, making it go faster and faster until it reaches speeds as high as $^3/_4$ of the speed of light!

3) **Put it where you want it:** The user of a sling has to let go at just the right point to cause the stone to go toward the target. In the cyclotron we release the ions by using a thin foil to strip the electrons off, switching the charge to positive and causing it to bend out of the cyclotron. There is one big advantage over the slingshot though… because the particles are still charged we can use fields to steer them to where we need them!

4) **See what happened:** Someone using a sling probably just looks to see if they hit their target, but the particles we work with are too small to see! So we need special detectors. We also use the protons for more complex purposes, often smashing them into nuclei to either see what is inside or to create a spray of new particles which we will then focus and direct to new experiments!

Lesson 1:   The Hydrogen Minus Ion

1.1 Overview

If we are going to make protons travel very fast, the first thing we need is to get protons. Fortunately nature supplies us with lots of these, all atoms contain protons after all. Most atoms are not JUST protons, of course. But the simplest atom, hydrogen, is a single proton with (usually) one electron paired with it. So the first ingredient in getting our protons is hydrogen!

The classical view of the atom imagines the electron "orbiting" the proton.
Something closer to the modern view can be pictured with the electron as smeared out into a "cloud" so that its position is not simply determined.
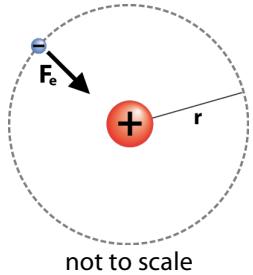
The classical diagram of the atom, as shown above, can be used to estimate the energy that it takes to remove the electron from the proton (the ionization energy). To do this we will start with the classical radius of the hydrogen atom, 5.29×10^{-11} m.

That seems pretty small (and it is!) but in fact this atom is almost entirely empty space. The best estimate of the radius of a proton is about 1.20×10^{-15} m, which is much, much smaller! To give you some idea, imagine that the atom were the size of the head of a pin. Put it in the middle of a football or soccer field. The electron would then be a speck of dust on the sidelines of the field.

Now take out everything else and just leave the pinhead and the dust-speck. Pretty empty isn't it?!

Let's look again at the diagram of the atom we have sketched. We can't call it a picture because the real ion is very small, and so can only be described in terms of quantum mechanics, and therefore is very weird indeed. But we can approximate things about it using a model like this. We have a radius as given above, and we know the charge on the electron and proton and their masses. So we can use what we know about force and circular motion to estimate the energy that must be supplied to separate the electron and proton in the hydrogen atom.
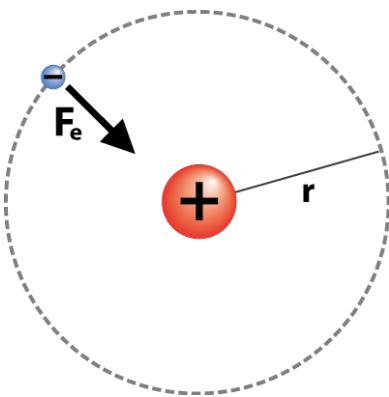
The charge between the electron and the proton causes a force attracting them together. The proton is almost 2000 times more massive than the electron, so we can treat this as the electron orbiting the proton. The inward force is given by Coulomb's law, while the inward acceleration is determined by the speed of the electron and the radius of the orbit. We can use these relationships to describe the properties of the orbit and its energy!

not to scale

To create the H- ions we need we must first blast the electron out of its 'orbit' around the proton. This creates separate protons and electrons. The protons and electrons are attracted to one another, so they will combine again. But sometimes two electrons will combine with one proton (creating a H- ion) and sometimes the proton will be left on its own (creating a H+ ion). We will then extract the H- ions (the H+ ions will eventually hit the walls of the chamber where they are created, and pick up an electron, which in turn will come from elsewhere until eventually everything evens out again!). To remove the electrons takes energy, though, because they are bound to the proton. High- energy electrons are used, and we can determine the energy needed (which is called the 'ionization energy'). We will guide you through how to calculate the ionization energy on the next page (it is also shown in the video!).

Worksheet 1.1   Finding the Ionization Energy of H

The questions below guide you through the calculations (which are also done for you in the video). Hints are given at the end of the worksheet.

Mass of electron:	9.11×10^{-31} kg
Charge on electron:	-1.60×10^{-19} C
Charge on proton:	$+1.60 \times 10^{-19}$ C
Radius of 'orbit':	5.29×10^{-11} m

1) If the atom were as we have shown in our model, what would the force be between the electron and the proton? In what direction?

Force is given by Coulomb's Law: $F_e = k \dfrac{q_1 q_2}{r^2}$

$$= 8.99 \times 10^9 \text{Nm}^2/\text{C}^2 \; \frac{(+1.60 x 10^{-19} C)(-1.60 x 10^{-19} C)}{(5.29 x 10^{-11} m)^2}$$

$$= -8.25 \times 10^{-8} \text{ N}$$

The sign indicates the force is attractive so the direction is inward.

(ans: 8.25×10^{-8} N inward)

2) If we imagine the electron in a classical circular orbit, what velocity would it have?

The force found in (1) is creating an inward (centripetal) acceleration.

F_e = Total Force = ma So $F_e = m \dfrac{v^2}{r}$

$$8.25 \times 10^{-8} \text{ N} = 9.11 \times 10^{-31} \text{kg} \; \frac{v^2}{5.29 x 10^{-11} m}$$

$$v = 2.19 \times 10^6 \text{ m/s}$$

(ans: 2.19×10^6 m/s)

3) What kinetic energy would it have?

$E_K = {}^1/_2$ m $v^2 = ({}^1/_2) \; 9.11 \times 10^{-31}$ kg $(2.19 \times 10^6$ m/s$)^2$

$\qquad\qquad = 2.18 \times 10^{-18}$ J

(ans: 2.18×10^{-18} J)

4) What electric potential energy would it have at this radius?

Potential energy for two charges separated by distance r is given by:

$$E_P = k\frac{q_1 q_2}{r} = 8.99\text{x}10^9 \text{ Nm}^2/\text{C}^2 \frac{(+1.60 x10^{-19}C)(-1.60 x10^{-19}C)}{5.29 x10^{-11}m} = -4.37\text{x}10^{-18} \text{ J}$$

The negative means energy is released when combining the two charges.

(ans: -4.37x10^{-18} J)

5) What is the total energy of the electron?

$$E_T = \text{Total energy} = E_K + E_P$$
$$= 2.18\text{x}10^{-18} \text{ J} + -4.37\text{x}10^{-18} \text{ J}$$
$$= -2.18\text{x}10^{-18} \text{ J}$$

*Note — the total energy is negative, indicating that the electron cannot leave (escape to zero potential energy at $r \rightarrow \infty$)
— total energy is exactly half the potential energy… this is a general result for circular motion in inverse square law fields and also applies for circular gravitational orbits

(ans: −2.18x10^{-18} J)

6) If this calculation were correct, how much additional energy would you have to add to remove one electron (remember, you need to give it a total energy ≥ 0 to escape, and it already has some kinetic energy)?

Clearly, if the total energy is −2.18x10^{-18} J, then the energy that must be added for the total energy to reach 0 is **+**2.18x10^{-18} J

(ans: 2.18x10^{-18} J)
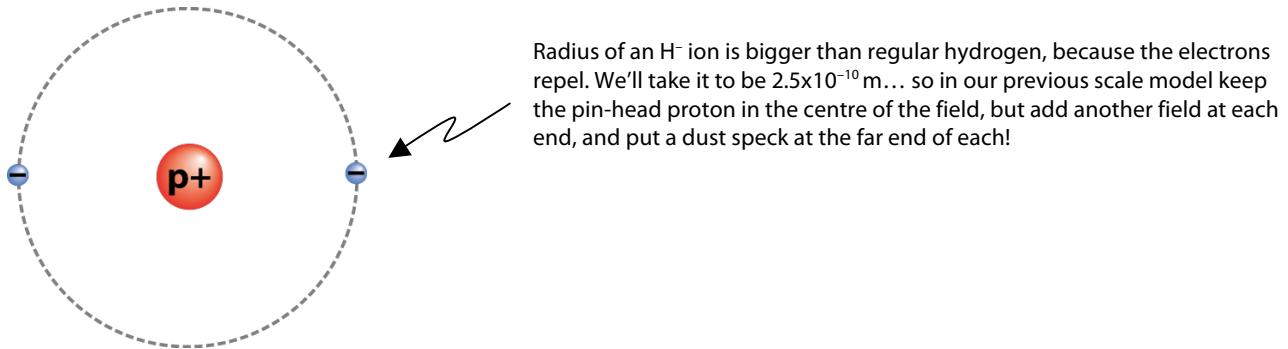
The real energy needed to remove an electron from an H atom is, in fact, 2.2x10^{-18} J (aka 13.6 eV)!

One should not necessarily be too impressed that we have gotten the right answer: the classical radius of the H atom, which we used for the questions, comes FROM this value!!
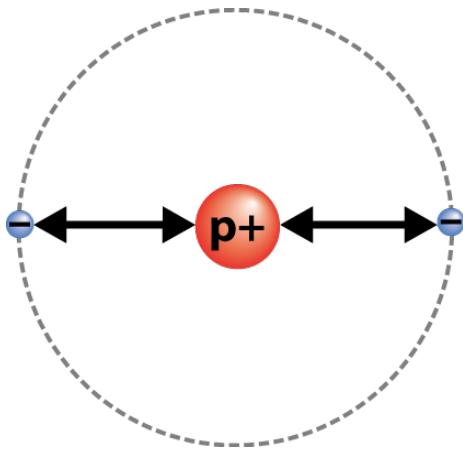
1.2 Extension: The H- Ion and its Ionization Energy

Many accelerators use protons, first removing the electron. But TRIUMF does something fiendishly clever, and uses hydrogen with an EXTRA electron instead. Later, by removing the electrons, we can turn these H⁻ ions into bare protons, and remove them from our cyclotron very easily!

Here is a sketch representing an H⁻ ion.

Radius of an H⁻ ion is bigger than regular hydrogen, because the electrons repel. We'll take it to be 2.5×10^{-10} m... so in our previous scale model keep the pin-head proton in the centre of the field, but add another field at each end, and put a dust speck at the far end of each!

The two are pictured on opposite sides of the proton because they repel each other, so they end up as far apart as possible. We will treat the electrons as at equal distance from the proton, and on opposite sides.
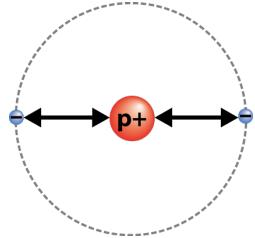
We found the ionization energy of the ordinary (neutral) hydrogen ion previously, let's see if we can extend the analysis to estimate the ionization energy of the H⁻ ion.

To do this we will need to keep in mind that all three charges exert forces on each other and all three store energy. We will focus on one of the electrons, treating it as being added to a neutral atom to create the H⁻ (which is more or less what happens most often).

The following worksheet guides you through this calculation.

Worksheet 1.2 Estimating the Ionization Energy of H⁻

Mass of electron:	9.11×10^{-31} kg
Charge on electron:	-1.60×10^{-19} C
Charge on proton:	$+1.60 \times 10^{-19}$ C
Radius of 'orbit':	2.5×10^{-10} m

1) If the atom were as we have shown in our model, what would the force be between one of the electrons and the proton? In what direction?

Force is given by Coulomb's Law: $F_e = k\dfrac{q_1 q_2}{r^2}$

$$= 8.99 \times 10^9 \text{Nm}^2/\text{C}^2 \; \frac{(+1.60 x 10^{-19} C)(-1.60 x 10^{-19} C)}{(2.5 x 10^{-10} m)^2}$$

$$= -3.70 \times 10^{-9} \text{ N}$$

The sign indicates the force is attractive so the direction is inward.

(ans: 3.7×10^{-9} N inward)

2) What would the force be between the two electrons? Does it attract or repel?

Force is given by Coulomb's Law: $F_e = k\dfrac{q_1 q_2}{r^2}$

$$= 8.99 \times 10^9 \text{Nm}^2/\text{C}^2 \; \frac{(-1.60 x 10^{-19} C)(-1.60 x 10^{-19} C)}{(2 x 2.5 x 10^{-10} m)^2}$$

$$= +9.24 \times 10^{-10} \text{ N}$$

The sign indicates the force is repelling so the direction is outward.

(ans: 9.2×10^{-10} N outward)

3) What is the total force acting on the electron?

Total force is the sum of these two forces, so
Total F = 3.70×10^{-9} N inward + 9.24×10^{-10} N outward = 2.77×10^{-9} N inward

(ans: 2.8×10^{-9} N inward)

4) If we imagine the electron in a classical circular orbit, what velocity would it have?

The force found in (3) is creating an inward (centripetal) acceleration.

F_e = Total Force = ma So $F_e = m\dfrac{v^2}{r}$

$$2.77 \times 10^{-8} \text{ N} = 9.11 \times 10^{-31} \text{ kg} \frac{v^2}{2.5 x 10^{-10} m}$$

$$v = 8.72 \times 10^5 \text{ m/s}$$

(ans: 8.7×10^5 m/s)

5) What kinetic energy would it have?

$$E_K = \frac{1}{2}mv^2 \;=\; (\frac{1}{2})\; 9.11\text{x}10^{-31} \text{ kg } (8.72\text{x}10^5 \text{ m/s})^2 = 3.46\text{x}10^{-19} \text{ J}$$

(ans: 3.46x10⁻¹⁹ J)

6) What total electric potential energy would the electron store in its relationship with the proton? (Technically this is stored in the field between the two, but we can treat it as being energy of the electron.)

Potential energy for two charges separated by distance r is given by: $E_P = k\dfrac{q_1 q_2}{r}$

$$= 8.99\text{x}10^9 \text{ Nm}^2/\text{C}^2 \; \frac{(+1.60 x 10^{-19} C)(-1.60 x 10^{-19} C)}{2.5 x 10^{-10} m}$$

$$= -9.24\text{x}10^{-19} \text{ J}$$

The negative sign means energy is released combining the two charges.

(ans: −9.24x10⁻¹⁹ J)

7) How much electric potential energy is stored by the interaction between the electron and the other electron? (Again, the energy is stored in the interaction but will be considered energy of the electron.)

Potential energy for two charges separated by distance r is given by: $E_P = k\dfrac{q_1 q_2}{r}$

$$= 8.99\text{x}10^9 \text{ Nm}^2/\text{C}^2 \; \frac{(-1.60 x 10^{-19} C)(-1.60 x 10^{-19} C)}{2 x 2.5 x 10^{-10} m}$$

$$= +4.62\text{x}10^{-19} \text{ J}$$

The positive sign means energy is stored pushing the two charges together.

(ans: +4.62x10⁻¹⁹ J)

8) What is the total potential energy of the electron?

Total potential energy = $E_{P+} + E_{P-}$ = −9.24x10⁻¹⁹ J + 4.62x10⁻¹⁹ J = −4.62x10⁻¹⁹ J

(ans: −4.62x10⁻¹⁹ J)

9) What is the total energy of the electron?

Total energy is just kinetic + potential = $E_K + E_P$ = 3.46x10⁻¹⁹ J + (−4.62x10⁻¹⁹ J) = −1.16x10⁻¹⁹ J
Note the total energy is negative, indicating that the electron cannot leave (escape to zero potential energy at $r \rightarrow \infty$).

(ans: −1.16x10⁻¹⁹ J)

10) If this calculation were correct, how much additional energy would you have to add to remove one electron?

To allow the electron to escape, its total energy must be made positive, which will take
+1.16x10⁻¹⁹ J.

(ans: 1.16x10⁻¹⁹ J)

The real energy needed to remove an electron from an H⁻ ion is 0.754 eV or 1.21x10⁻¹⁹ J. Our value is within 5% of the accepted value – impressively close given that we did such a rough calculation, but don't be too impressed! Why is the fact that we are so close not really so amazing?
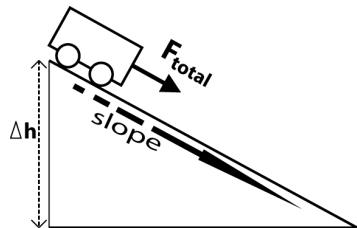
Our estimate is within 5% of the correct value, but again this is not really as strong as it might seem because we chose the radius from a range of possible values (atoms being not so well defined things), and the choice made was partly determined by the ionization energy!

Lesson 2: Initial Acceleration

2.1 Overview
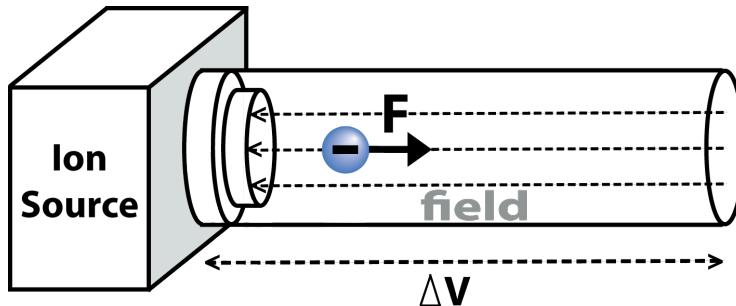
Once we have created the ions we need to get them into the cyclotron. That means we need to give them a "kick" to start them on their way (a little acceleration to get them to the big accelerator!). To do this we use an electric potential difference. The whole ion source sits inside a box that is at a negative potential difference to the outside. As it crosses this potential difference each ion is accelerated to a relatively high velocity.

The electrical potential difference works very much the same way a gravitational potential difference does for an object travelling down a frictionless ramp:

A cart on a ramp accelerates down because of a force **(F)** caused by the **slope**, which is itself caused by the height difference **(Δh)**. You could find the speed using force and acceleration, or much more easily by looking at potential energy becoming kinetic.

You may have experienced gravitational fields creating acceleration and making things go fast in just this way if you have ever been skiing, or for that matter sledding, ridden go-carts (of the old, roll-down-a-hill kind) or coasted on your bike down a hill!

A charge in the ion source accelerates "down" because of a force **(F)** caused by the **field**, which is itself caused by the potential difference **(ΔV)**. You could find the speed using force and acceleration, or much more easily by looking at potential energy becoming kinetic. (Notice that the negative charge goes opposite to the field, one confusing thing about negative charges!)
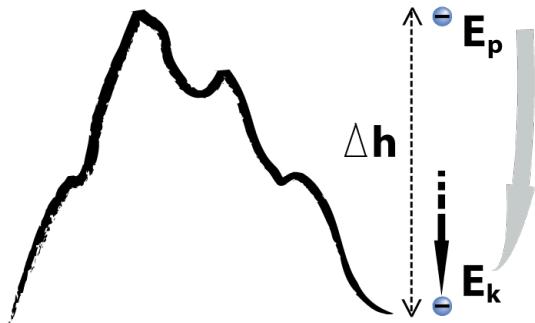
Although you may not have realized it, you have probably encountered electric fields being used to accelerate charges in just this way inside your TV or computer monitor (well, not the flat screen ones, but the older CRTs). A surprising number of other devices use similar methods too!

So, why do we choose to use the electric field to make our protons go quickly? In theory we could use gravity to accelerate them, but gravity is just too weak for that in the vicinity of the earth. Near a black hole it is different, and particles near black holes do reach incredible speeds because of gravity! Since we don't have any black holes close by (and a good thing too!) we will have to use another kind of field – electric fields – that we can make very strong.

In the worksheets we will follow the calculations of the video to guide you through looking at energy changes for gravitational fields and electrostatic fields (such as the ones used at TRIUMF).

Worksheet 2.1   Finding the Initial Speed of the H⁻ Ions

The questions below guide you through the calculations (which are also done for you in the video).

Height of Mt. Everest:	8,848 m
Earth's gravitational field:	9.8 N/kg down
Mass of H⁻:	1.67×10^{-27} kg
Speed of light (c):	3.00×10^8 m/s

1) Suppose we tried to accelerate a H⁻ ion by dropping it from the height of Mt. Everest. What potential energy would it have when at the top of the mountain (relative to sea level)?

$E_P = mg\Delta h = (1.67 \times 10^{-27}$ kg$)(9.8$ N/kg$)(8848$ m$) = 1.45 \times 10^{-22}$ Nm $= 1.45 \times 10^{-22}$ J

(ans: 1.45×10^{-22} J)

2) If this potential energy is all converted to kinetic energy, then what velocity will the proton have?

$E_K = E_P$

$^1/_2 mv^2 = 1.45 \times 10^{-22}$ J

$v = \sqrt{\dfrac{1.45 x 10^{-22} \, kgm^2 s^2}{\frac{1}{2}(1.67 x 10^{-27} \, kg)}} = 416$ m/s
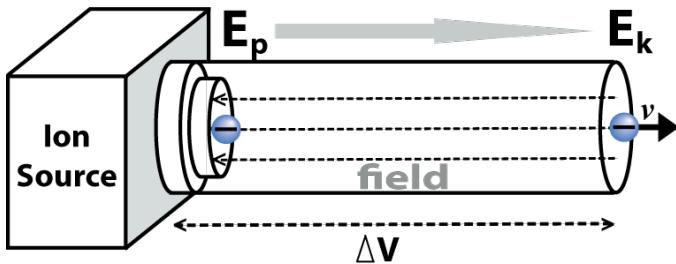
(ans: 416 m/s)

3) What is that as a percentage of light speed (the speed of light is about 3.00×10^8 m/s)?

Percentage of light speed $= \dfrac{416 m/s}{3.00 x 10^8 m/s} = 0.00014\%$

(ans: 0.00014%)

We eventually want to accelerate our protons up to around $^3/_4$ of the speed of light. Clearly gravity isn't going to do it! Let's try that with an electric field!

Mass of H⁻:	1.67×10^{-27} kg
Charge on H⁻:	-1.60×10^{-19} C
ΔV:	300 000 V
Speed of light (c):	3.00×10^{8} m/s

We can maintain a potential difference of 300 000 V between our Ion Source (where we make the H⁻ ions) and the outside. This 300 000 V takes the place of the height difference.

4) Using the relationship for electrostatic potential energy ($E_P = q\Delta V$) and kinetic energy ($E_K = \frac{1}{2} mv^2$), find the speed of an H⁻ ion that crosses this potential difference.

$$E_K = E_P$$
$$\tfrac{1}{2}\, mv^2 = q\Delta V$$
$$v = \sqrt{\frac{(-1.60 x 10^{-19}C)(300000 J/C)}{\tfrac{1}{2}(1.67 x 10^{-27} kg)}} = 7.59 \times 10^6 \text{ m/s}$$

(ans: 7.59×10^6 m/s)

**5) What is this as a percentage of light speed?**
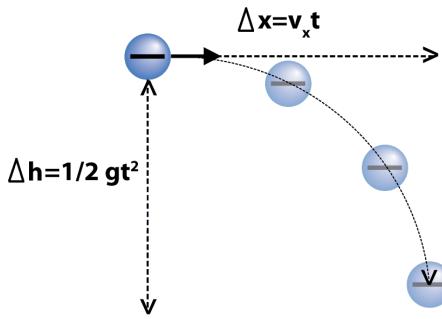
$$\text{Percentage of light speed} = \frac{7.59 x 10^6 m/s}{3.00 x 10^8 m/s} = 2.5\%$$

(ans: 2.5 %)

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2.2 Extension: Forces Affecting the Beam

The beam is then brought to the centre of the cyclotron. This means that the ions have to travel close to about 100 m before they reach the point where we bend them down into the cyclotron. That's quite a ways, and we might worry about how far the ions will fall in that time (after all, gravity still affects them, just as it affects every mass!). Fortunately this is not an issue… there just isn't time for them to fall!

We know that all masses in a gravitational field are accelerated downward by the field, regardless of their motion. A mass thrown sideways falls at the same rate as one with no sideways motion, so the high velocity of the H⁻ ions does not change their falling.
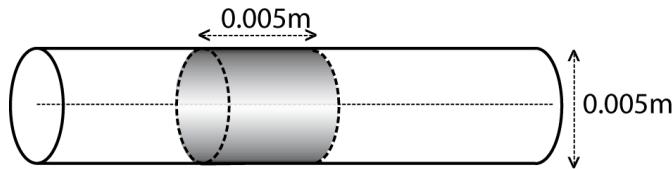

We can apply this ballistic motion idea to the H⁻ ions just as we can to falling balls that are thrown. Since the ion is moving very quickly we will see that it does not have enough time to fall very far on its way to the cyclotron.
Gravity is not the only force acting on the ion though! It is part of a beam of ions, and all the ions are negatively charged. Since like charges repel, these ions all repel each other. Figuring out the real effect of this is mathematically complicated because the spreading out depends on the force and the force depends on the spreading out! The math for dealing with this is not actually that hard, but most high school students won't have learned it yet, so we will dodge it and only approximate the effect.

First we need to have some idea what sort of field an ion near the edge of the beam will experience because of the other ions (we'll look at the edge because that is where the greatest spreading out is). The ions are in a beam (a line of charges), so it isn't just a simple matter of point charges. For those students who have not studied the field from a line of charge, though, we can approximate the line using a point charge.

Either way we'll start by finding out how much charge there is in a section of the beam:
The beam is a cylinder of charge about 5 mm across. A typical strength of the high intensity beam from the ion source is about 6 mA = 0.0006 C/s. So we will start by finding out how much charge there is in a 5 mm section of the beam:

Given the speed of the beam we can calculate the amount of charge in a 5 mm section:

Charge in a given length = Charge per time x (time to go the length)
$$q = I (L/v)$$
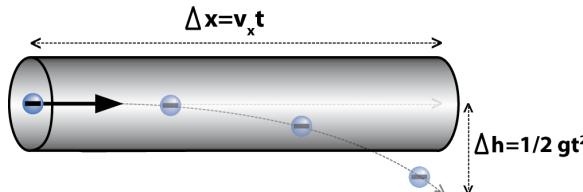
If we are familiar with the field for a line of charge we could use this charge to find the field and the force on a charge at the edge of the beam right away. If not we can use Coulomb's law to approximate the force between this much charge and an H⁻ ion at the edge of the beam.

Using that force we can estimate the acceleration and get a very rough idea of how much the beam will spread in the time it is travelling.

Worksheet 2.2   Getting the Beam to the Cyclotron

The questions below look at how fields affect the beam as it travels to the cyclotron.

$\Delta x = v_x t$

$\Delta h = 1/2\ gt^2$

Speed of the beam:	7.59×10^6 m/s
Distance travelled:	100 m
Gravitational field:	9.8 N/kg down

1) How long does it take the beam to travel the distance to the cyclotron?

$$\Delta t = \frac{\Delta x}{v_x} = \frac{100 m}{7.59 x 10^6 m/s} = 1.32 \text{x} 10^{-5} \text{ s}$$

(ans: $1.32 \text{x} 10^{-5}$ s)
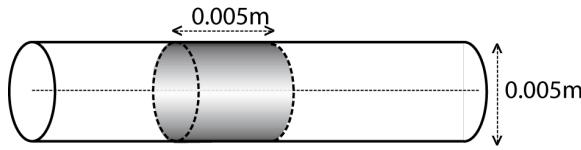
2) How far will an object fall in this time?

$\Delta y = \frac{1}{2}\ g\Delta t^2 = 1/2\ (9.8\ \text{m/s}^2)(1.32 \text{x} 10^{-5}\ \text{s})^2 = 8.51 \text{x} 10^{-10}\text{m}$

(ans: $8.51 \text{x} 10^{-10}$ m)

Clearly this is not far enough to worry about (remember we used $2.5 \text{x} 10^{-10}$ m as the radius of H⁻, so this is less than 2 atoms width!

What about the repulsion of all those negative ions though? Electric forces are usually very big compared to gravity, is that true here?

0.005m

0.005m

Beam current:	−0.006 C/s
Speed of the beam:	7.59×10^6 m/s
Radius of the beam:	0.0025 m
Mass of H⁻:	$1.67 \text{x} 10^{-27}$ kg
Charge on H⁻:	$-1.60 \text{x} 10^{-19}$ C

3) How long does it take the beam to go 5.0 mm?

$$\Delta t = \frac{\Delta x}{v_x} = \frac{0.005 m}{7.59 x 10^6 m/s} = 6.59 \text{x} 10^{-10} \text{ s}$$

(ans: $6.59 \text{x} 10^{-10}$ s)

4) How much charge travels along that section of beam?

$\Delta q = I\Delta t = (-0.006\ \text{C/s})(6.59 \text{x} 10^{-10}\ \text{s}) = -3.95 \text{x} 10^{-12}$ C

(ans: $-3.95 \text{x} 10^{-12}$ C)

5) If we treat the charge of this section of the beam as like a point charge at the centre of the beam, then what is the force between the charge of the beam and a H⁻ ion near the beam's edge?
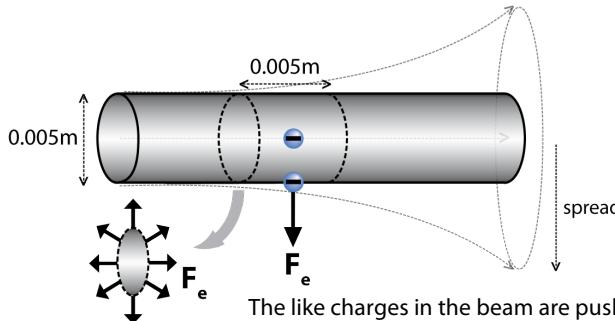
r = radius of the beam = 0.0025 m, so

$$F = k\frac{q_1 q_2}{r^2} = 8.99 \text{x} 10^9\ \text{Nm}^2/\text{C}^2\ \frac{(-3.95 x 10^{-12} C)(-1.60 x 10^{-19} C)}{(0.0025 m)^2}$$

$= +9.12 \text{x} 10^{-16}$ N (positive means it's a repelling force)

(ans: $9.12 \text{x} 10^{-16}$ N, repelling)

The force of the ions repelling each other that you just found is going to push the ions apart and spread out the beam:

0.005m

0.005m

spread

F_e

F_e

Repelling force (5):	9.12×10^{-15} N
Mass of H⁻:	1.67×10^{-27} kg
Distance travelled:	100 m
Time to cyclotron:	1.32×10^{-5} s

The like charges in the beam are pushing each other apart, so they will accelerate away from each other, making the beam spread out just like the force of gravity makes the beam fall.

6) What acceleration will the H— ion initially have because of the repulsive force?

$$\sum F = ma \qquad \text{so } a = F_e/m$$
$$= \frac{9.12 x10^{-15} N}{1.67 x10^{-27} kg} = 5.46 \times 10^{11} \text{ m/s}^2 \text{ (outward)}$$

(ans: 5.46×10^{11} m/s²)

7) How far would the H— ion be pushed if that acceleration lasted for the whole time it was travelling to the cyclotron?

$$\Delta y = \frac{1}{2} a\Delta t^2 = \frac{1}{2} (5.46 \times 10^{11} \text{m/s}^2)(1.32 \times 10^{-5} \text{s})^2 = 47 \text{ m}$$

(ans: 47 m)

Wow, that's a lot of spread! Of course it is an over-estimate… it will really be much less because as the charges get further apart the force will get a lot smaller. But even if the spread were only a small fraction of that we'd be in very serious trouble… our beam would be tearing itself apart before it got anywhere. We need a way to push it back together or to *focus* it. We'll look at that in the additional materials for the next lesson! If you have studied the field created by a line of charge, then you can calculate the force using this result. If you have not studied that then ignore the next three questions!

8) Look back at the description of the beam, and the calculation of the amount of charge in a 5 cm long section. The beam is essentially a very long cylinder of charge, which we can treat as a line of charge along its axis. Use your result from (4) to find the linear charge density of the beam.

$$\lambda = \frac{q}{L} = \frac{-3.95 x10^{-12} C}{0.005m} = -7.91 \times 10^{-10} \text{ C/m}$$

(ans: 7.91×10^{-10} C/m)

9) Now we can find the field from the line of charge at the edge of the beam. The distance from the center of the beam is our radial distance, so what is the field there?

$$E = \frac{2k\lambda}{r} \left\{ OR \frac{\lambda}{2\pi\varepsilon_0 r} \right\}$$
$$= \frac{2(8.99 x10^9 Nm^2/C^2)(7.91 x10^{-10} C/m)}{0.0025m} = 5700 \text{ N/C}$$

(ans: 5700 N/C)
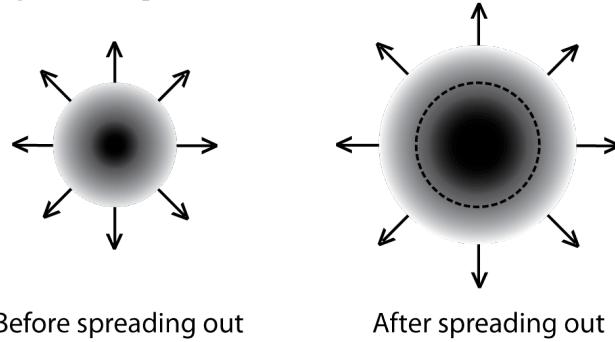
10) What is the force on an H⁻ ion at this point due to this field?

$$F = qE = (-1.60 \times 10^{-19} \text{ C})(5700 \text{ N/C}) = 9.12 \times 10^{-16} \text{ N}$$

(ans: 9.12×10^{-16} N)

2.3 Extension: Electrostatic Focusing
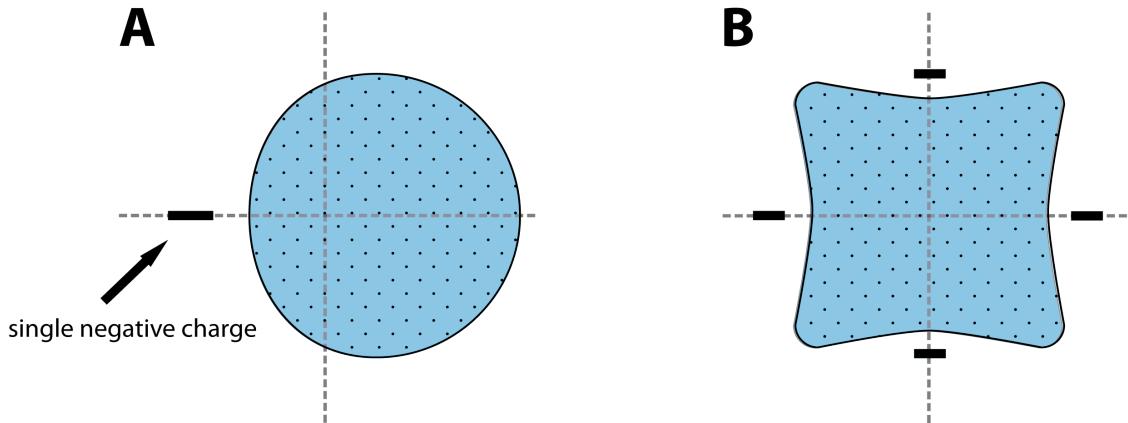
As the beam travels from the unit that gives it its initial acceleration to the cyclotron, the negative charges in the beam will repel one another, causing them to spread out.

Before spreading out   After spreading out
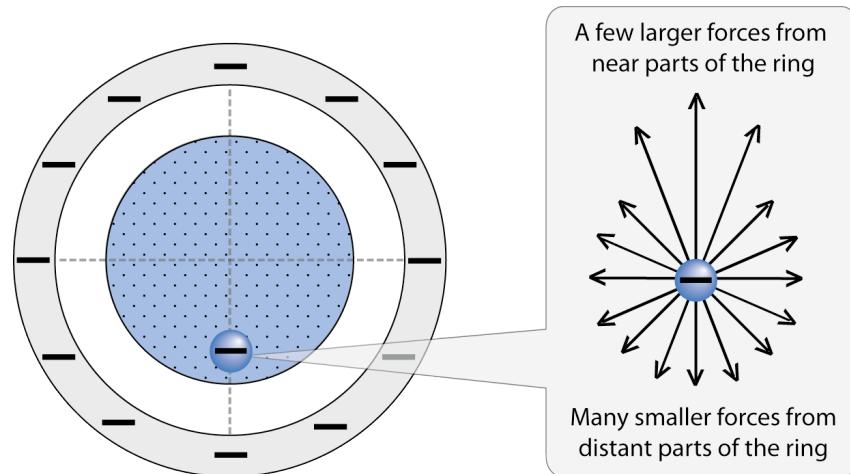
In the previous extension we saw that while gravity was too weak to have much effect in the time the beam was travelling, the electrostatic force was very much greater. Because of this, the spread on the way to the cyclotron would be large, possibly metres, so clearly we DO have to worry about this… therefore, we need to focus the beam.

The most obvious thing to do is to use negative charges to push the beam together. This won't work, however, and for an interesting reason.

Imagine the beam in cross section (the shape we see if we cut across the beam) as we showed it above. We've assumed that the beam is roughly circular. If we use a single negative charge we would just push the charges of the beam to one side, though compressing the beam slightly because ions near the charge are more affected than those further out (**A**). If we try to surround the charge with a ring of charges, though, they start to cancel each other out (**B**)!

15
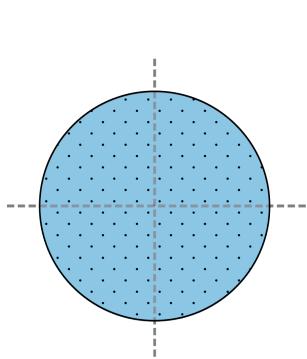
With a complete ring the forces all cancel out, and the ring does nothing (there is no net field inside)! This is why a conductor with charge spread over the outside has no field within it, or the principle of a Faraday cage if one thinks about it!

A few larger forces from near parts of the ring

Many smaller forces from distant parts of the ring
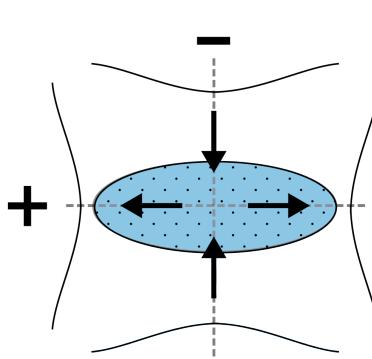
Total Force on an Ion=0

To avoid this, we use what is called a *quadrupole* to focus our beam with electric fields. A quadrupole has two + and two – charges. It compresses the beam in one direction and stretches it in the other. By using two, we can produce a net focusing of the beam.

first vertical focussing   followed by horizontal focussing
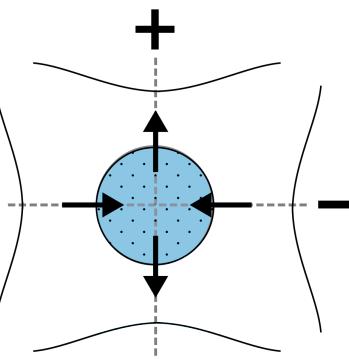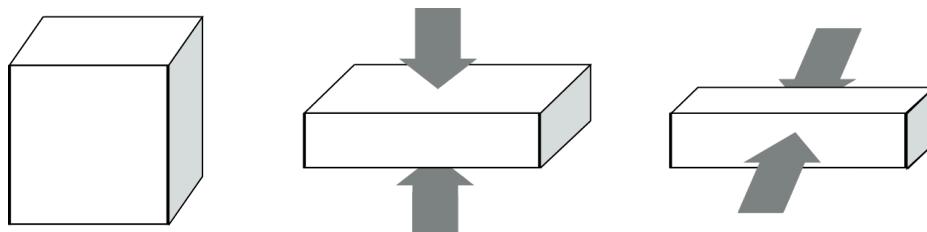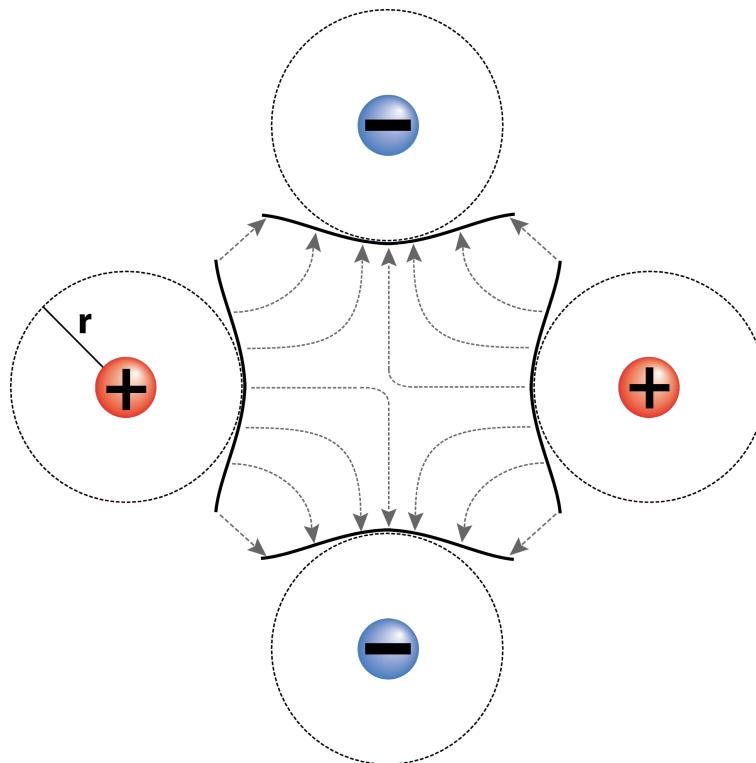
(1)   **(2)**   **(3)**

This is like taking a piece of plasticine and squishing it first one way, and then the other!

Worksheet 2.3 Electrostatic Focusing

1) The diagram below shows a sketch of an electric quadrupole. The positive and negative poles of a quadrupole are very similar to point charges in their fields, at least outside of the plates (they are actually cylinders but we'll treat them as spheres for now). The picture shows a quadrupole with the imaginary point charges at their centres. Sketch the electric field inside the quadrupole.

2) Consider a quadrupole like the one shown below. One of the negative plates (we call it a 'pole', like a magnet's pole) can be treated like a charge of r = 2.5 cm radius. The potential on this surface is 10,000 V. What charge at the centre would produce this potential at 2.5 cm distance?
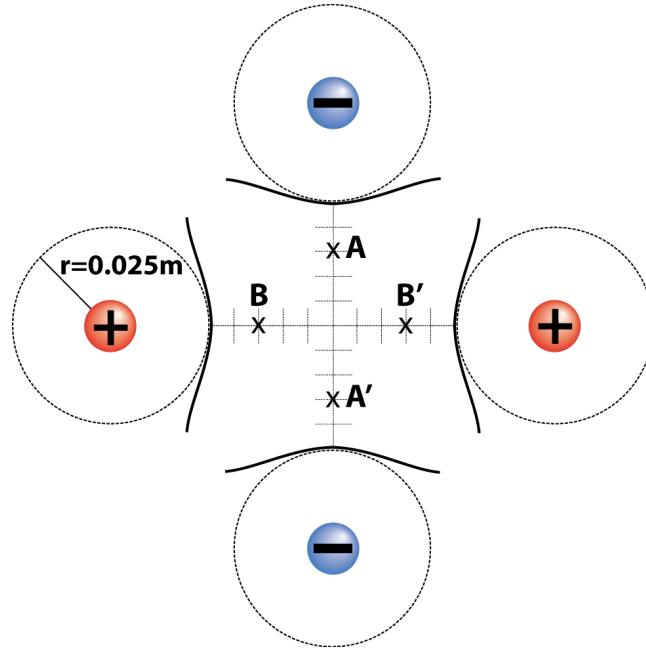
$$V = k\frac{q}{r} \text{ so } q = \frac{rV}{k} = \frac{(0.025m)(10000Nm/C)}{(8.99x10^{10}Nm^2/C^2)}$$
$$= 2.78x10^{-8} \text{ C}$$

(ans: 2.78x10⁻⁸ C)

3) Treating the quadrupole plates as such point charges, estimate the force on an H⁻ ion at points A and B, and by symmetry at A' and B'. Each mark on the axis is 0.005 m.

At **A**: F = (Force from close – charge) + (Force from farther – charge) + (Force from the two + charges)

close – charge:
$$F = k\frac{q_1 q_2}{r^2} = 8.99x10^9 Nm^2/C^2 \frac{(-2.78x10^{-8}C)(-1.60x10^{-19}C)}{(0.035m)^2}$$
$$= 3.27x10^{-14} \text{ N} \downarrow$$

far – charge:
$$F = k\frac{q_1 q_2}{r^2} = 8.99x10^9 Nm^2/C^2 \frac{(-2.78x10^{-8}C)(-1.60x10^{-19}C)}{(0.065m)^2}$$
$$= 9.47x10^{-15} \text{ N} \uparrow$$

+ charges:
$$r = \sqrt{(0.050m)^2 + (0.015m)^2} = 0.0522m$$
$$F = k\frac{q_1 q_2}{r^2} = 8.99x10^9 Nm^2/C^2 \frac{(-2.78x10^{-8}C)(-1.60x10^{-19}C)}{(0.0522m)^2}$$
$$= 1.47x10^{-14} \text{ N} \uparrow$$

Components in the ↔ direction clearly cancel.

Vertically F_Y = F sin θ = F$\left(\frac{0.015m}{0.0522m}\right)$ = 4.22x10⁻¹⁵ N↓   x 2 = 8.44x10⁻¹⁵ N↓

Total F_A = 3.27x10⁻¹⁴ N↓ + 9.47x10⁻¹⁵ N↑ + 8.44x10⁻¹⁵ N↓ = 3.17x10⁻¹⁴ N↓

F_A = 3.17x10⁻¹⁴ N↓	$F_{A'}$ = 3.17x10⁻¹⁴ N↑
F_B = 3.17x10⁻¹⁴ N←	$F_{B'}$ = 3.17x10⁻¹⁴ N→

4) Find the acceleration at each of the points in the previous problem (A, A', B, and B').

F= ma so a = F/m = 3.17x10^{-14} N/1.67x10^{-27}kg = 1.90x10^{13} m/s^2

(ans: 1.90x10^{13} m/s^2)

5) The quadrupoles are effectively about 15 cm long. How long are the ions affected by this field?

Δt = Δx/v$_X$ = 0.15 m/7.59x10^6 m/s = 1.98x10^{-8} s

(ans: 1.98x10^{-8} s)

6) We will pretend that the field as constant for the particles (which we know it is not, but the time is short enough that they don't move sideways too much, so it's on to make this approximation). What inward velocity is given to the H$^-$ ions as they pass through the quadrupole?

Δv$_Y$ = aΔt = (1.90x10^{13} m/s^2)(1.98x10^{-8} s) = 3.75x10^5 m/s
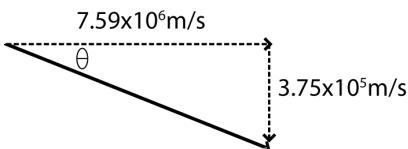
(ans: 3.75x10^5 m/s)

Note: you can use Δy=1/2 aΔt^2 to check the statement that the sideways motion is small. If you do so you will get a displacement of about 4 mm, which is relatively small but not completely trivial at this scale.

7) What angle will the new velocity make with the original velocity (assumed to be sideways)?
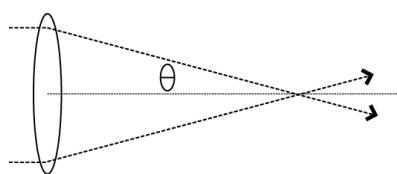
v$_X$ is not affected, so our new velocity is at an angle:

7.59x10^{6}m/s

3.75x10^{5}m/s

$$Tan\theta = \frac{3.75x10^5\,m/s}{7.59x10^6\,m/s} = 0.003703 \qquad \theta = 2.83°$$

(ans: 2.83°)

8) How far must they travel before they would hit the centre (this is called the 'focal length'). Remember that A (and the other points) are 0.015 m from the centre of the beam.

$$Tan\theta = \frac{3.75x10^5\,m/s}{7.59x10^6\,m/s} = 0.003703 = \frac{\Delta y}{\Delta x}$$

$$\Delta x = \frac{0.015}{0.003703} = 0.30m$$

(ans: 0.30 m)
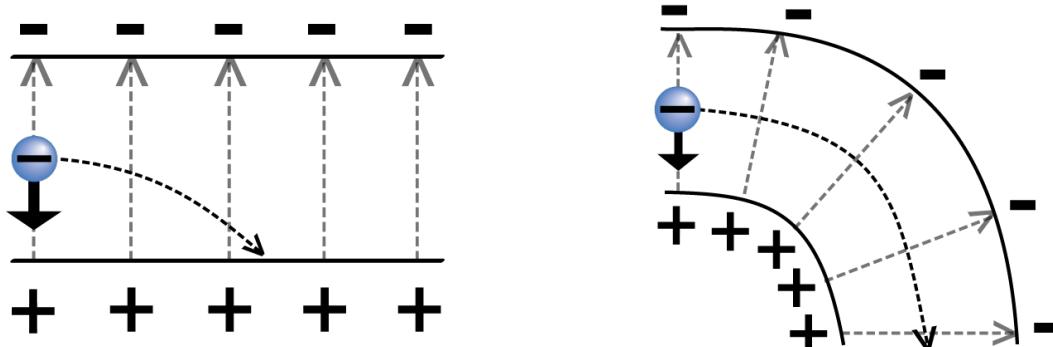
The actual focal length of the electrostatic focusing unit is about 0.25 m, so we are fairly close given our rather rough estimation (especially of the effective charges). Some of the other parameters of the quadrupoles were adjusted slightly to compensate for this in these calculations.

Lesson 3:   Electrostatic Beam Steering
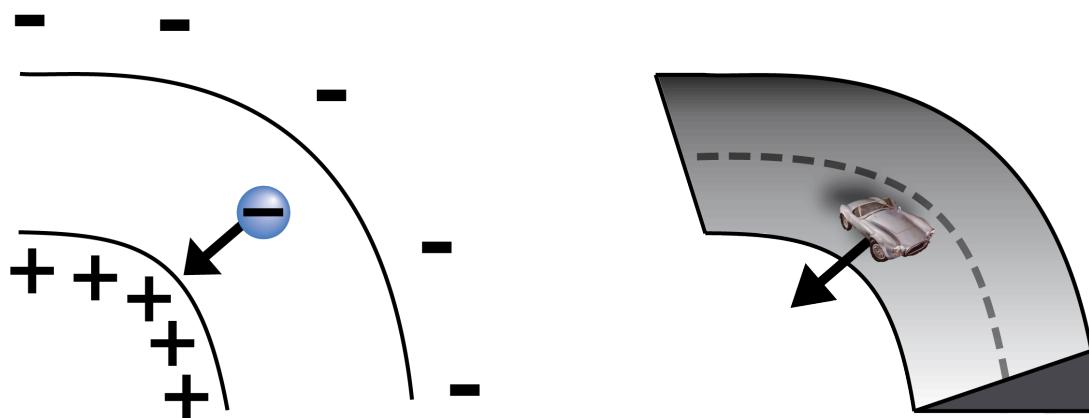
3.1 Overview

Before getting to the cyclotron we have to bend the beam down 90°. But making the charges in the beam go around the corner means we will need to push them inward to create an inward acceleration (centripetal acceleration). This is done using electrostatic fields.

To create a uniform field we use parallel plates. If the plates are flat, it is easy to see that the beam will simply curve into the positive plate. So what we want to do is curve the plates the same way the beam is curving. The effect is to create a uniform inward force on the beam.

The bending caused by the curved plates is just like the inward force from a banked curve (in fact the potential acts just like the height of the curve, and the field like the slope). So we can imagine the ions going around the curve like racecars!
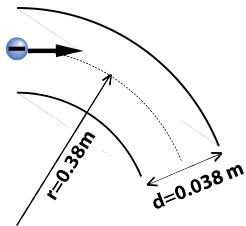
The actual bending is done in two 45° pieces (partly so that the beam can be focused in between, partly to take a short cut)!

In the worksheet you will calculate the potential difference in the bender (the calculations are also done in the video).

Worksheet 3.1 Electrostatic Beam Steering

The questions below guide you through a calculation of the potential needed to steer the electrons down into the cyclotron. Consider a particle passing between the curved plates of an electrostatic bender. The schematic below shows an example of this and some of the measurements from the benders at TRIUMF:

Mass of H⁻:	1.67×10^{-27} kg
Charge on H⁻:	-1.60×10^{-19} C
Speed of the beam:	7.59×10^{6} m/s
Radius of curvature:	0.38 m
Separation of the plates:	0.038 m

1) What is the inward (centripetal) acceleration of a H⁻ ion if it is following the curve of the bender?

$$a_c = \frac{v^2}{r} = \frac{(7.59 x 10^6 \, m/s)^2}{0.38 m} = 1.51 x 10^{14} \, m/s^2$$

(ans: 1.51×10^{14} m/s² inward)

2) What inward force is needed to cause this much acceleration?

$F_c = ma_c = (1.67 \times 10^{-27} \text{ kg})(1.51 \times 10^{14} \, m/s^2) = 2.52 \times 10^{-13}$ N

(ans: 2.52×10^{-13} N inward)

3) What field would create that force?

$$E = \frac{F}{q} = \frac{2.52 x 10^{-13} N (inward)}{-1.60 x 10^{-19} C} = 1.57 x 10^6 \text{ N/C outward}$$

(ans: 1.57×10^6 N/C outward)

4) What potential difference must there be across the plates to create that field?

$$E = \frac{\Delta V}{d} \qquad \text{so } \Delta V = Ed = (1.57 x 10^6 \text{ N/C})(0.038m) = 5.98 x 10^5 \text{ J/C}$$

(ans: 5.98×10^5 V)

The actual potential across these bending plates is 60 000 V, so the calculation is extremely close!

5) Now let's try this more symbolically. Using the equations you drew on in questions 2-5, create an equation for the potential difference in terms of mass, charge, inward acceleration, and separation of the plates.

$$\Delta V = Ed, \qquad E = \frac{F}{q} \qquad \text{so } \Delta V = \frac{F}{q}d, \quad F = ma_c \qquad \text{so } \Delta V = \frac{ma_c d}{q}$$

(ans: $\Delta V = \dfrac{ma_c d}{q}$)

6) Use this to solve for the potential difference across the plates.

$$\Delta V = \frac{ma_c d}{q} = \frac{(1.67 x 10^{-27} kg)(1.51 x 10^{14} \, m/s^2)(0.038m)}{1.60 x 10^{-19} C} = 5.98 x 10^5 \text{ V}$$

(ans: 5.98×10^5 V)
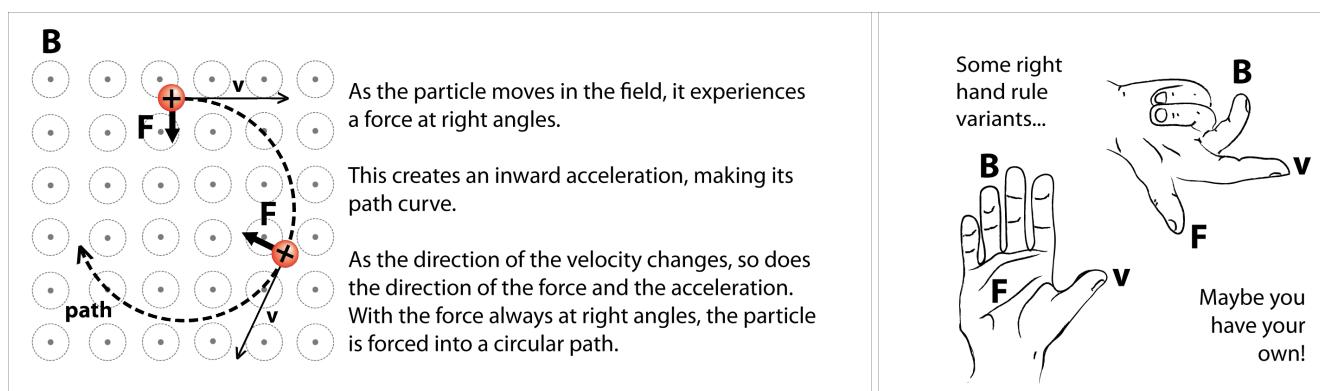
The video proceeds directly from finding the inward acceleration to finding the potential by combining equations. We have separated steps here to provide additional guidance to students having trouble working directly with symbols.

Lesson 4:   Circular Motion in a Magnetic Field - The Cyclotron

4.1 Overview
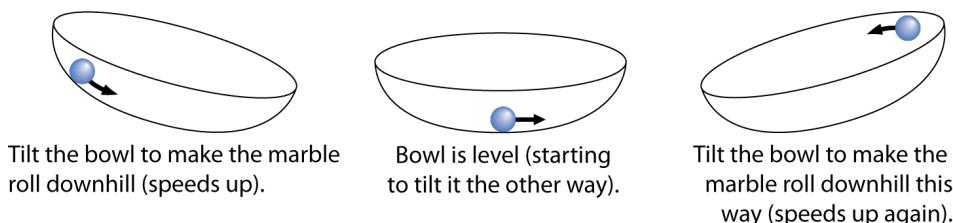
At last, we have gotten our ions safely to the cyclotron! Now the REAL business of accelerating the ions begins! To do this we need to add energy to the ions, just as we did with the first acceleration. But this is not easy. We need to add a LOT of energy. Doing that in one shot would be very difficult, requiring voltages that are not feasible. We can show, for example, that it would take a potential difference of over 500 million volts to get our ions up to $^3/_4$ of the speed of light. So we have to do it in stages, boosting the energy by a manageable amount each time.

To get the required energy we'll need thousands of such boosts. We could have a very long series of such boosting voltages (a linear accelerator) or we could bend the particles around so they go in a circle, and accelerate them a little bit each time around. And that's just what a cyclotron does! As you have probably studied, the secret behind the cyclotron is that, a charge moving in a magnetic field follows a circular path. This is because the force created by a magnetic field is always at right angles to the particle's velocity. You probably learned this using some form of the right hand rule.

As the particle moves in the field, it experiences a force at right angles.

This creates an inward acceleration, making its path curve.

As the direction of the velocity changes, so does the direction of the force and the acceleration. With the force always at right angles, the particle is forced into a circular path.

Some right hand rule variants…

Maybe you have your own!

Inside the cyclotron is a powerful magnetic field, which keeps the ions going in a circular path around and around the cyclotron. Of course we still have negative H ions, so we would need to reverse the direction or use a "left hand rule"!
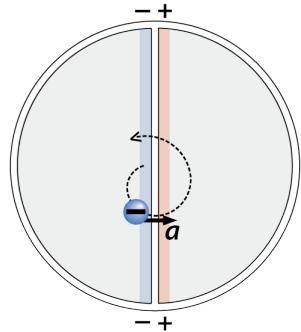
We then arrange "plates" in the cyclotron to give the particle a 'boost' each time it goes around. The idea is rather like the way you might roll a marble around inside a bowl, tilting the bowl to make it go faster and faster.

Tilt the bowl to make the marble roll downhill (speeds up).

Bowl is level (starting to tilt it the other way).

Tilt the bowl to make the marble roll downhill this way (speeds up again).
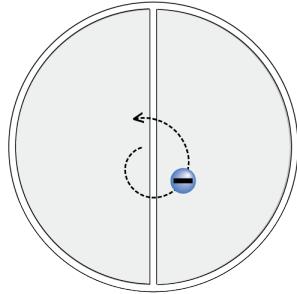
Notice that as the marble roles faster and faster it will roll higher and higher around the curve of the bowl, automatically rolling higher until greater radius and the steeper slope of the bowl provide the right balance of inward force to keep it going in a circle. Eventually it will be going so fast that it flies out of the bowl!

With the marble, we used gravitational potential to make it roll faster, and we changed the tilt (raised first one side then the other) to make it always roll "downhill".
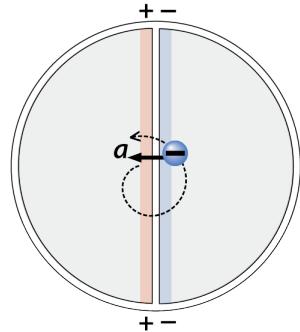
In the cyclotron we aren't dealing with a rolling marble of course, but with ions; the way to make ions go faster is to use an electric potential difference. So the cyclotron has two "D" shaped regions inside, across which a potential difference is applied:

Make the "D"s + and − so that the electron speeds up crossing the gap.
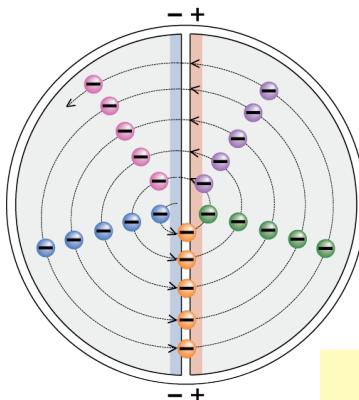
Resetting the "D"s... no gap so this doesn't change anything.
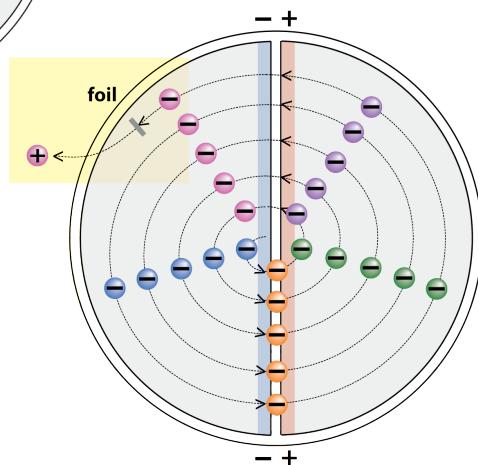
Make the "D"s + and − the other way so that the electron speeds up again.

So every time they go around, the ions get TWO extra energy boosts! With each "boost" the energy goes up, the speed goes up, and the radius goes up and the ions go around a little further out. But amazingly enough they don't change the amount of time they take! That is one of the things that makes it possible for the cyclotron to produce a continuous stream of charges. In the worksheet and in the calculations of the video you can work out the *cyclotron frequency*, which tells us how often the particles make a complete circuit. By switching every time they come around, we keep getting them going faster and faster.

In fact, TRIUMF is even cleverer than just using this! By switching the Ds 5 times in a period rather than just once, TRIUMF can have 5 separate bunches of ions in each 'circle' all accelerating at once:
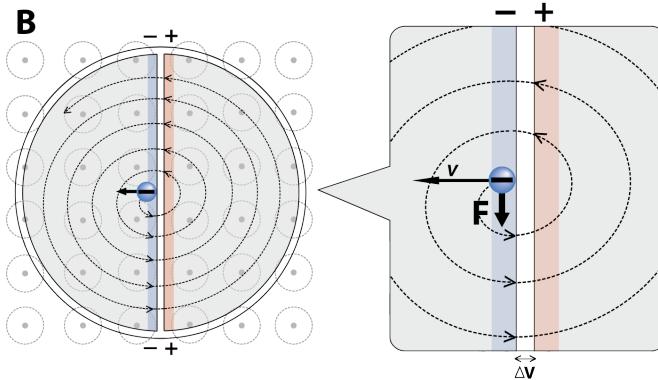
We will eventually get our ions out of the cyclotron in a continuous series of pulses, but how do we get the ions OUT of the cyclotron? Here is where using H⁻ pays off, because it gives us a really easy way to extract the protons… we just get rid of the electrons! The charge then goes from –1 to +1, and the proton will curve the opposite way, straight out of the cyclotron! To do this we use a very thin foil (about 0.025 mm thick). The proton is not much affected, but the electrons are so light that they are stripped right off.

foil

- Beam hits foil
- Electrons are removed
- H⁻ becomes H⁺
- Positive charge → curves the other way
- H⁺ ions (protons) leave the cyclotron

By putting the foil at different points, we can extract the protons at any of a wide range of energies.

Worksheet 4.1   Circular Motion in a Magnetic Field - The Cyclotron

Mass of H⁻:	1.67×10^{-27} kg
Charge on H⁻:	-1.60×10^{-19} C
Initial velocity:	7.59×10^{6} m/s
Magnetic field:	0.30 T
ΔV between "D"s:	186000 V

1) What is the initial force on the H⁻ ion due to the magnetic field?

$F = qvB = (1.60 \times 10^{-19} \text{ C})(7.59 \times 10^{6} \text{ m/s})(0.30 \text{ T}) = 3.64 \times 10^{-13}$ N

(ans: 3.64×10^{-13} N)

2) What inward acceleration does this force cause?

$F = ma,$ so $a = \dfrac{F}{m} = \dfrac{3.64 \times 10^{-13} N}{1.67 \times 10^{-27} kg} = 2.18 \times 10^{14}$ m/s²

(ans: 2.18×10^{14} m/s²)

3) What radius is the circle that the ion would trace out if nothing else affected it?

$F_C = ma_C = m\dfrac{v^2}{r}$ so $r = \dfrac{v^2}{a} = \dfrac{(7.59 \times 10^{6} m/s)^2}{2.18 \times 10^{14} m/s^2} = 0.264$ m

(ans: 0.264 m)

4) Combine the equations from the calculations of (1) to (3) to create a formula for finding the radius directly from the mass, charge, velocity, and magnetic field.

$F = ma$

$qvB = m\dfrac{v^2}{r}$

$r = \dfrac{mv}{qB}$

(ans: r = $\dfrac{mv}{qB}$)

5) How long would it take to make one full circle at this speed? Remember that the circumference of circle is 2πr.

$T = \dfrac{C}{v} = \dfrac{2\pi r}{v} = \dfrac{2\pi(0.264 m)}{7.59 \times 10^{6} m/s} = 2.19 \times 10^{-7}$ s

(ans: 2.19×10^{-7} s)

6) What is the initial kinetic energy of the ion?

$E_K = {}^{1}/_{2} mv^2 = {}^{1}/_{2}(1.67 \times 10^{-27} \text{ kg})(7.59 \times 10^{6} \text{ m/s})^2 = 4.81 \times 10^{-14}$ J

(ans: 4.81×10^{-14} J)

7) When the ion crosses the gap between the "D"s, how much energy does it gain?

$\Delta E_P = q\Delta V = (1.60 \times 10^{-19} \text{ C})(186000 \text{ V}) = 2.98 \times 10^{-14}$ J

(ans: 2.98×10^{-14} J)

8) What is its new kinetic energy?

$E_{Kf} = E_{Ki} + \Delta E_P = 4.81 \times 10^{-14} \text{ J} + 2.98 \times 10^{-14} \text{ J} = 7.79 \times 10^{-14} \text{ J}$

(ans: 7.79x10⁻¹⁴ J)

9) What is the new velocity of the H⁻ ion?

$E_K = {}^1/_2 \, mv^2$ so $v = \sqrt{\dfrac{2E_K}{m}} = \sqrt{\dfrac{2(7.79 x 10^{-14} J)}{1.67 x 10^{-27} kg}} = 9.66 \times 10^6 \text{ m/s}$

(ans: 9.66x10⁶ m/s)

10) What is the radius of its circular path now?

Using our previous equation:

$r = \dfrac{mv}{qB} = \dfrac{(1.67 x 10^{-27} kg)(9.66 x 10^6 m/s)}{(1.60 x 10^{-19} C)(0.30T)} = 0.336 \text{ m}$

(ans: 0.336 m)

11) How much time does it take for one full circle?

$T = \dfrac{C}{v} = \dfrac{2\pi r}{v} = \dfrac{2\pi(0.336m)}{9.66 x 10^6 m/s} = 2.19 \times 10^{-7} \text{ s}$

(ans: 2.19x10⁻⁷ s)

Notice that the time is *exactly the same* (except, perhaps, for rounding if you rounded your answers). This is extremely useful, because it means that we can have many "batches" of ions in the machine at one time! Although the ion is going faster, it also makes a larger circle so it has father to go. These effects cancel out and the time for one orbit is constant.

12) Combine the equation for r which you found in (4) with the calculation of time from the circumference to get an equation for the period.

$T = \dfrac{C}{v} = \dfrac{2\pi}{v}\left(\dfrac{mv}{qB}\right) = \dfrac{2\pi m}{qB}$

(ans: $T = \dfrac{2\pi m}{qB}$)

Notice that the velocity cancels out, so no matter what the velocity is, the ions have the same period. This period also tells us the cyclotron frequency, and that tells us how often we need to switch the voltage on the "D"s to keep the ions flowing!

13) Using your previous results for the period, calculate the cyclotron frequency.

$f = \dfrac{1}{T} = \dfrac{1}{2.19 x 10^{-7} s} = 4.57 \times 10^6 \text{ Hz}$

(ans: 4.57x10⁶ Hz)

14) Express the cyclotron frequency symbolically in terms of the ion's mass and charge, and the magnetic field.

$f = \dfrac{1}{T} = \dfrac{1}{2\pi m/qB} = \dfrac{qB}{2\pi m}$

(ans: $f = \dfrac{qB}{2\pi m}$)

4.2 Finding the Cyclotron Frequency from Experimental Data

We saw before that we can use a thin carbon foil to remove the electrons from our H⁻ ion, turning it into a H⁺ ion and causing it to bend out of the cyclotron. By moving the foil we can cause this to happen at different radii of the cyclotron, so we can get particles at a wide range of energies/velocities. We can use actual data from the beam to confirm the cyclotron frequency we found from the magnetic field and the ion's charge and mass.

Recall that the period of the cyclotron was found from the speed and the circumference of the orbit:

$$v = \frac{C}{T} = \frac{2\pi r}{T}$$

$\frac{1}{T}$ is the frequency, so $v = \frac{2\pi r}{T} = 2\pi r \frac{1}{T} = 2\pi r f$

If we write this as $v = (2\pi f)r$, we see that velocity is a linear function of the radius – meaning that if we plot velocity against radius we will get a straight line.

$$v = (2\pi f) \quad r$$

$(\rightarrow \quad y = (slope) \quad x)$

So plotting a graph with velocity on the y-axis and radius on the x-axis should give us a straight line with a slope equal to $2\pi f$, which lets us find the real cyclotron frequency directly from data.
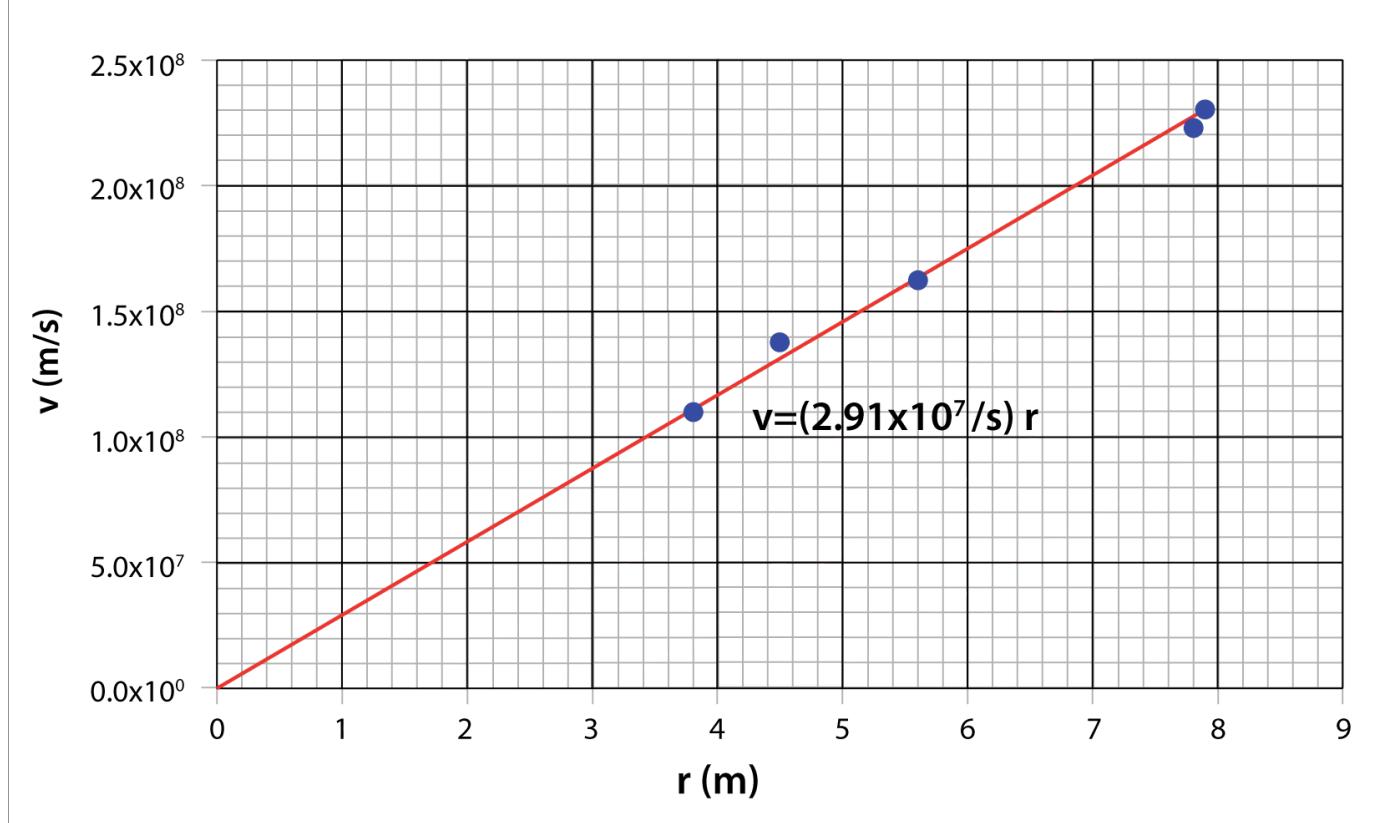
Here are some measured values from TRIUMF, showing the velocity of the particles extracted at different radii:

r (m)	v (m/s)
3.81	1.10×10^8
4.50	1.37×10^8
5.59	1.63×10^8
7.80	2.25×10^8
7.92	2.30×10^8

You can plot this data on the following page.

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Worksheet 4.2   Finding the Cyclotron Frequency from Experimental Data

Beam Velocity at Various Extraction Radii

1) Draw a best-fit line for your data points.

2) Find the slope of your best-fit line.

(ans: approximately 2.9×10^7 /s)

3) Use this slope to determine the cyclotron frequency.

v = $(2\pi f)$r,   so slope = $2\pi f$

$$f = \frac{2.9 \times 10^7 /s}{2\pi} = 4.6 \times 10^6 /s$$

(ans: 4.6×10^6 Hz)

This is almost exactly the frequency we calculated (4.57×10^6 Hz), so we see that our high school physics really does describe this most important cyclotron parameter!

Lesson 5: Electromagnetic Beam Steering

5.1 Overview
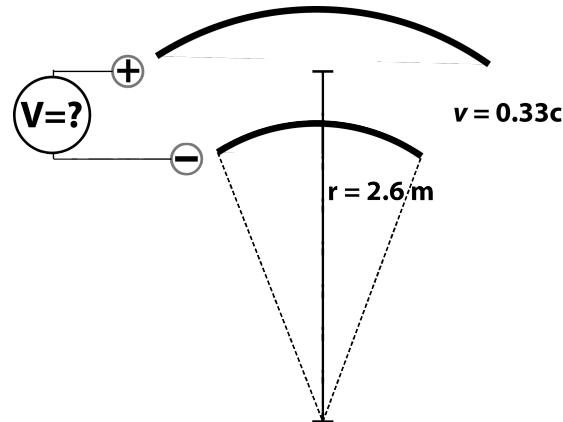
Now that we have extracted the protons out of the cyclotron, they must be *steered* and kept in the beam pipe by *focusing* as they are delivered to their experiment. Recall that the negatively-charged hydrogen ions also were steered and focused as they were transported into the cyclotron through the injection line. However, the protons exiting the cyclotron travel from 33% to 75% the speed of light, compared with about 2.5% the speed of light for the negative hydrogen ions. The high speeds of the protons require different steering and focusing techniques than those used in the injection line.

Worksheet 5.1 Electrostatic Steering

Let's first consider what it would take to steer a proton beam *exiting* the cyclotron with electrostatic plates similar to those used to steer the negative hydrogen ions *into* the cyclotron. This calculation is identical to the one in Lesson 3, so refer to that worksheet for more details.

So let's calculate the potential difference across electrostatic plates required to steer a proton beam exiting the cyclotron.

Mass of proton:	1.67×10^{-27} kg
Charge on proton:	1.60×10^{-19} C
Proton velocity:	1.00×10^{8} m/s
Radius of curvature:	2.6 m
Δd between plates:	0.038 m

1) What is the centripetal acceleration of the protons following the curve?

$$a_c = \frac{v^2}{r} = \frac{(1.00 \times 10^8 \, m/s)^2}{2.6 m} = 3.85 \times 10^{15} \text{ m/s}^2$$

(ans: 3.85×10^{15} m/s²)

2) What inward force is required to cause that acceleration?

$F_c = ma_c = (1.67 \times 10^{-27} \text{ kg})(3.85 \times 10^{15} \text{ m/s}^2) = 6.42 \times 10^{-12}$ N

(ans: 6.42×10^{-12} N)

3) What field would create that force?

$F_c = F_e = qE = 6.42 \times 10^{-12}$ N Therefore $E = \frac{6.42 \times 10^{-12} N}{1.60 \times 10^{-19} C} = 4.01 \times 10^7$ V/m

(ans: 4.01×10^7 V/m)
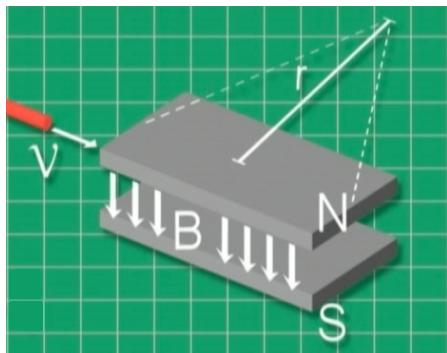
4) What potential difference across the plates will create that field?

$\Delta V = Ed = (4.01\text{x}10^7\ \text{V/m})(0.038\ \text{m}) = 1.52\text{x}10^6\ \text{V}$

(ans: 1.52x10⁶ V)

The electrical potential difference across a 3.8 cm gap required to steer a proton with v=0.33c around a 2.6 m radius of curvature turns out to be an astounding **1.5 MILLION volts**! If that sounds like a lot, it is, even for nuclear physicists. It is very difficult to create that kind of constant voltage. The image at right shows an early particle accelerator (so-called "Cockroft-Walton") that wouldn't be out of place in a science-fiction movie — it generated a mere million volts! Even if one could create 1.5 million volts across a 3.8 cm gap, it would not be possible to prevent it from arcing and sparking.

So another solution is required to steer protons travelling a significant fraction of the speed of light.

One Million Volt Particle Accelerator 1937

Worksheet 5.2 Electromagnetic Steering

Fortunately, there is a solution. Recall that the force on a particle of charge q and velocity v by a magnetic field perpendicular to v is given by F = qvB. This so-called "Lorentz force" is perpendicular to both v and B, following the right-hand rule. So let's calculate what B would be to steer the proton beam as in the previous section.

Mass of proton:	1.67×10^{-27} kg
Charge on proton:	1.60×10^{-19} C
Proton velocity:	1.00×10^{8} m/s
Centripetal Force needed:	6.42×10^{8} N
(found previously)	

1) What vertical magnetic field (perpendicular to proton's direction of motion) would create that force?

$F_B = F_C$

$qvB = 6.42\text{x}10^{-12}\ \text{N}$ Therefore $B = \dfrac{6.42x10^{-12}N}{(1.60x10^{-19}C)(1.00x10^{8}m/s)} = 0.40\ \text{T}$

(ans: 0.40 T)

5.2 Extension: Designing the Dipole Magnet

Vertical magnetic fields of up to 1 Tesla are relatively straightforward to design with normal magnets. So, let's design one, namely, the magnet from the previous exercise.

Magnets come in two types: **permanent**, like the ones sticking notes up on your refrigerator, and **electromagnets**, which use a current-carrying coil of wire to create the magnetic field. Permanent magnets are usually too weak and also not adjustable, so TRIUMF uses electromagnets almost exclusively.
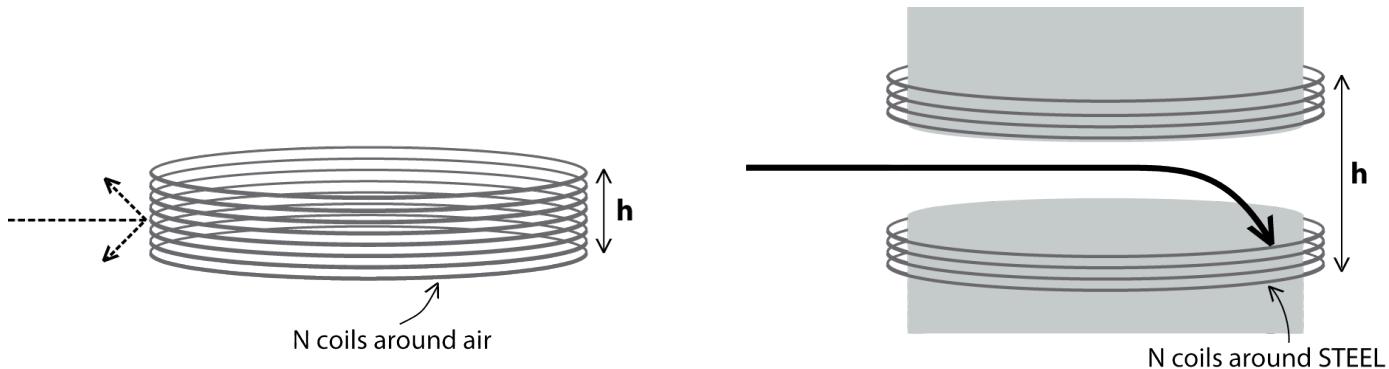
The formula for the magnetic field inside a coil of current-carrying wire is: $B = \mu \dfrac{N}{h} i$

N coils around air

h

where:

- B is the magnetic field strength (Tesla)
- i is the current in the wire coil (Amperes)
- N is the total number of loops
- h is the total coil height (metres)
- μ is the magnetic constant ($4\pi \times 10^{-7}$ Tm/A in air)

HOWEVER, a proton beam cannot pass *through* the coils of the air solenoid, so instead a gap must be created to allow the protons to pass freely through the magnet, with the field-generating coils above and below the gap. Refer to the above figure. IF the coils wrap around steel rather than air, which effectively increases the magnetic field in the steel, it turns out that the magnetic field across the gap of height h is *almost exactly the same* as the above expression for an air coil of height h. That is one reason why iron is used to make electromagnets.

h

N coils around air
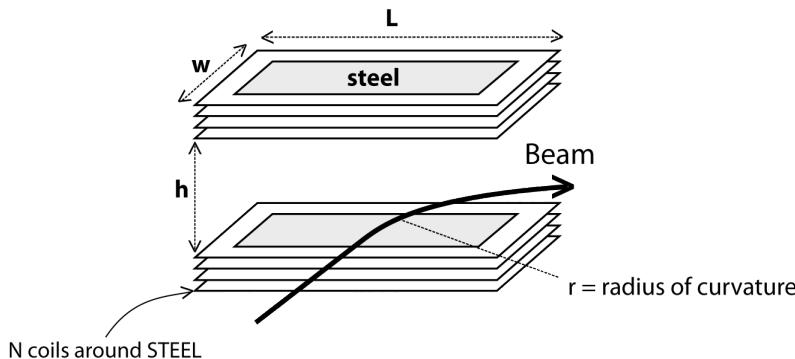
Beam can't get through!

h

N coils around STEEL

Beam is bent as needed!

In the worksheet you can explore how a magnet is designed to produce the field needed. This is an interesting thing because there are many possible magnets, so choosing the right one is not automatic but requires balancing other factors.

Worksheet 5.3   Designing a Steering Magnet

We can now use the formula for the magnetic field ($B = \mu \dfrac{N}{h} i$) to design our steering magnet.

We'll keep the proton velocity and the magnet radius of curvature the same as before (v = 0.33 c and r = 2.6 m).

Typical electromagnets at TRIUMF are made of roughly rectangular segments, using copper wires wrapped around iron cores of width w = 0.5 m and length L = 1.25 m, and an air gap of h = 10 cm (4"), so we'll assume those values for our magnet

**1) We can choose the current and the number of coils. If we use 2 coils (one above the gap and one below) what current is needed?**

$$B = \mu \frac{N}{h} i \text{ so } i = \frac{Bh}{\mu N} = \frac{(0.4T)(0.10m)}{(4\pi \times 10^{-7} Tm/A)(2)} = 1.59 \times 10^4 \text{ A}$$

(ans: 1.59 x 10⁴ A)

That's almost 16,000 A, which is a LOT! The wires to carry that much current without melting would have to be massive… which is possible but maybe a better design can be found.

2) If we design our magnet to need only 1.0 A of current then how many coils would we need?

$$B = \mu \frac{N}{h} i \text{ so } N = \frac{Bh}{\mu i} = \frac{(0.4T)(0.10m)}{(4\pi \times 10^{-7} Tm/A)(1.0A)} = 3.18 \times 10^4 \text{ coils}$$

(ans: 3.18 x 10⁴ coils)

That's a lot of coils!

3) What length of wire would be needed for 31800 coils?

Each coil is the perimeter of a rectangle with w = 0.5 m and L = 1.25 m = 2(0.5 m) + 2(1.25 m) = 3.5 m
so 31800 coils is 31800 x 3.5 m = 1.11 x 10⁵ m

(ans: 1.11 x 10⁵ m)

That's over 111 kilometers of wire! That will take up a lot of space even if we make the wire quite thin (and we can't make it too thin or won't carry 1.0 A safely… that's still a pretty significant current). Clearly the preferred answer should be somewhere in between.

Alert problem solvers will notice that although N and i can vary, the product N i is a constant: $Ni = \dfrac{Bh}{\mu}$

The actual N and i chosen depend on practical issues such as how hard to make the wire is (like the very thick wires you'd need to carry 16000 A), how much space it will take up when wound (111 km of wire, for example!), resistance of the wire (thicker wire has less resistance, but if you put a huge current through it that won't help) cooling requirements (because there is still resistance and a lot of heat created) and cost.

Addendum: TRIUMF's Preferred Solution

At TRIUMF, a steering magnet at the exit of the cyclotron has the following characteristics:

- Field: **B** = 1.07 Tesla (to be able to steer protons up to v = 0.75 c)

- Gap: **h** = 0.10 m

- Coil length: **C** = 2.74 m (average)

- **N** ·**i** = 90835 amp·turns

Note that a more accurate formula for **N** ·**i** was used, but the difference is small, about 5% (90835 versus 86510 using our formula).

The TRIUMF engineers decided to use **N** = 84 and **i** = 540 Amperes, using a *square* copper wire 1.31 cm on the side, with a 0.73 cm diameter *hole* in the middle (refer to figure on the right)! The current is large, but not so big that the wires become too difficult to make and handle, and easy enough for several larger standard wires to bring in the current from the outside to the magnet.

Some labs use circular wire but TRIUMF engineers prefer square since they stack together better around the coil. For this magnet, the coils are wrapped 12 high and 7 deep around the magnet poles (refer to bottom figure). Coolant is pumped through the hole at high pressure to dissipate the heat generated by the resistance of the wire.

Some additional questions:

1) **To produce 0.40 T (as needed for our original magnet) using this number of coils, what current is needed?**

$$B = \mu \frac{N}{h} i \quad \text{so} \quad i = \frac{Bh}{\mu N} = \frac{(0.4T)(0.10m)}{(4\pi \times 10^{-7} Tm/A)(168)} = 189 \text{ A}$$
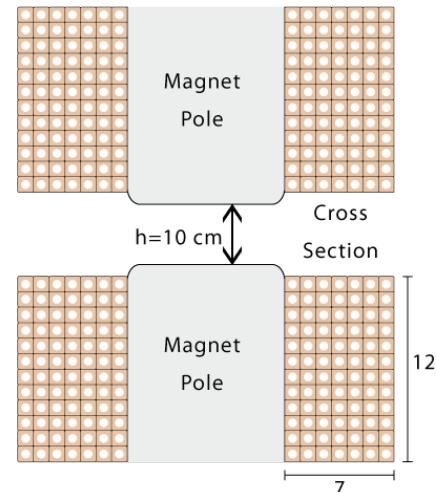
(ans: i = 189 A)

2) **If you have studied resistivity use the cross sectional area and length of wire to calculate the resistance of the wire.**

$$R = \rho \frac{L}{A} = 1.72 \times 10^{-8} \frac{460m}{(0.0131m)^2 - \pi \cdot (0.00365m)^2} = 0.061 \, \Omega$$

(ans: R = 0.061 Ω)

3) **Given the resistance found in (2) how much heat energy is produced in the coils every second?**

P = I²R = (189A)² (0.061 Ω) = 2200 W

(ans: P = 2200 W)

4) **Remembering that the energy absorbed by a coolant such as water is given by ΔE = mc_pΔT, and using c_p for water = 4180 J/KgC, estimate how many kg (litres) of water must flow through the coils each second to keep them cool with no more than a 40 ºC increase in water temperature?**

ΔE = mc_pΔT   so in each second 2200 J = m(4180 J/kgºC)(40ºC)

which gives m = 0.013 $P = \frac{\Delta E}{\Delta t} = mc_p$ so m = P/c_p = 2200 W/4180 J/kgºC = 0.013 kg = 13 g/s (13 mL/s)

(ans: 13 g/s [13 mL/s])

5.3 Extension: Engineering in Designing the Dipole Magnet

More advanced engineering problem: Once you've designed your magnet using the above formula, think a bit about the fact that in real life you'll be using *real* copper wires that have a non-zero diameter and resistance, and what that might do to influence your design.

There are many issues for a magnet engineer to consider, and certainly students cannot be expected to be knowledgeable of them, so here we just want to give them a taste of the design process. First, we need to know the number of coils and their current. Plugging the parameters into the formula:

$$N \cdot i = g\frac{B}{\mu_o} = 0.1\frac{0.4}{4\pi x10^{-7}} = 31,831 \text{ amp} \cdot \text{turns}$$

Therefore, any combination of **N** and **i** multiplying to ≥ 31831 would work. e.g. 2 coils (one above and one below the gap) running 15916 amps, or even 31832 (15916 each above and below) coils running 1 amp.

But which combination do we choose? There isn't one answer, but one must consider:

- Power loss in the wires: wires have a small resistance, and current flow will cause power loss

- Space: wires have a non-zero thickness, and the coils need to fit in the available space. Also, the power loss through the wires generates heat, which must be dissipated away by: e.g. blowing cold air in the spaces, or with cold water pipes, both taking up more space.

- Manufacturing: how hard will it be to make such a magnet?

To complicate matters further, copper wire of a given diameter has guidelines governing the maximum amount of current that it can carry safely. The table at the right gives examples of resistance and maximum recommended currents for copper wires, based on a maximum recommended voltage loss of $V_L = 0.0485$ Volts/m (recall, V=IR).

Copper wire resistance and maximum current
(source: http://www.powerstream.com/Wire_Size.htm)

AWG Gauge	Diameter [mm]	Ohms per m	Max current [A]
0000	11.68	0.00016	302.00
0	8.25	0.00032	150.00
3	5.83	0.00064	75.00
6	4.11	0.00130	37.00
9	2.91	0.00260	19.00
12	2.05	0.00521	9.30
15	1.45	0.01044	4.70
18	1.02	0.02094	2.30
21	0.72	0.04198	1.20
24	0.51	0.08420	0.58

So let's see what we have:

- Each coil has circumference C = 2L+2W = 3.5 m

- The maximum recommended wire voltage loss (0.0485 Volts/m) and the **N** · **i** value 31831 amp·turns together mean the power loss is 5400 Watts regardless of the exact value of **N** and **i**:

$$\frac{P}{NC} = V_L i \qquad \text{but } i = \frac{31831}{N}$$

$$\text{so } P = N(3.5m)(0.0485V/m)(\frac{31831}{N} amp \cdot turns) \cong 5400 \text{ Watts}$$

That is comparable to an electric stove element, so cooling the wires will be a priority! Now let's see what this means for the two extreme situations:

1) N=2, i=15916 amps: This huge current is beyond the table, but doubling the wire diameter increases the maximum current by 4x, so we estimate the wire diameter to be 9 cm! A wire that thick is tough to make and tougher to handle, and moreover difficult to cool throughout. There would be plenty of space though.

2) N=31832, i=1 amp: To accommodate 1 amp, the wire should be AWG-21 (refer to table above), or about 0.72 mm in diameter. The total wire length would be 31832x3.5 m = 111412 m long (over 111 km!). Wrapping 111 km of wire is a challenge, and space must be made to allow for coiling the wires, so space will be an issue.

Clearly the preferred answer should be somewhere in between.

5.4 Extension: TRIUMF's Preferred Solution

At TRIUMF, a steering magnet at the exit of the cyclotron has the following characteristics:

- Field: **B** = 1.07 Tesla (to be able to steer protons up to v = 0.75c)
- Gap: **h** = 0.10 m
- Coil length: **C** = 2.74 m (average)
- **N** ·**i** = 90835 amp·turns

Note that a more accurate formula for **N** ·**i** was used, but the difference is small, about 5% (90835 versus 86510 using our formula).

The TRIUMF engineers decided to use **N**=84 and **i**=540 Amperes, using a *square* copper wire 1.31 cm on the side, with a 0.73 cm diameter *hole* in the middle (refer to figure on the right)! The current is large, but not so big that the wires become too difficult to make and handle, and easy enough for several larger standard wires to bring in the current from the outside to the magnet.

Some labs use circular wire but TRIUMF engineers prefer square since they stack together better around the coil. For this magnet, the coils are wrapped 12 high and 7 deep around the magnet poles (refer to bottom figure). Coolant is pumped through the hole at high pressure to dissipate the heat. Using water as a coolant, the coolant temperature rises about 40°C at the maximum from the heat loss in the wires!

5.5 Extension: How a Quadrupole Magnet Focuses Charged Particle Beams

A quadrupole electromagnet has four magnetic poles, two north and two south, oriented as a cross so that the like poles face each other, like the figure on the right. Quadrupole magnets are used to *focus* charged particle beams, unlike dipole magnets that *steer* these beams. This problem will walk you through a graphical approach to understanding how a quadrupole works.

First, what do we want a focusing element to do? Let's consider 3 examples of a ray of a charged particle beam, or light, approaching a focusing element:

a) a ray approaching along the axis of the element has no need to be focused, so it should proceed straight through the element

b) a ray little bit off axis needs to be steered back towards the axis only a small amount to bring it back into focus

c) while a ray far off axis must be steered a lot to bring it back into focus
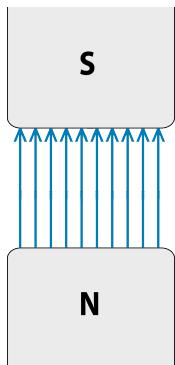
We see that a focusing element should leave the straight rays alone, and bend rays away from the axis back toward the axis proportional to their distance away from the axis.   Since a magnetic field is used to steer charged particles, then this field must be *zero* on the axis, growing progressively stronger away from the axis.

Let's see how this is done in a magnetic quadrupole by drawing the magnetic field lines on the figure in the worksheet, and determining their effect on a proton beam passing through the quadrupole (moving into the page in the figure).
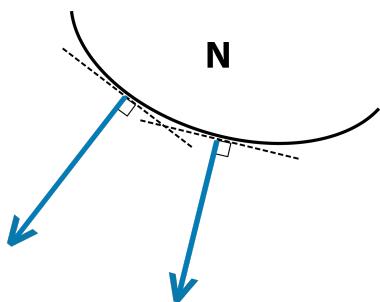
Magnet Basics

First, we need a few basic facts about how field lines behave in a magnet.
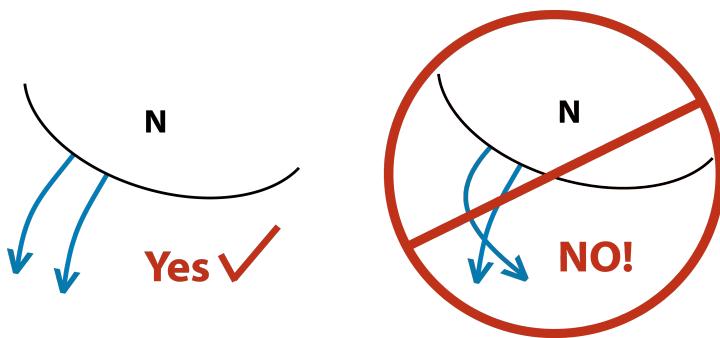
1. Magnetic field lines go from the North pole toward the South pole.

2. Field lines *always* exit or enter the surface of the magnet perpendicular to the surface. If the surface is curved, they enter/exit perpendicular to the tangent line to the surface.

3. Field lines *never* cross each other! Adjacent lines are not necessarily parallel.
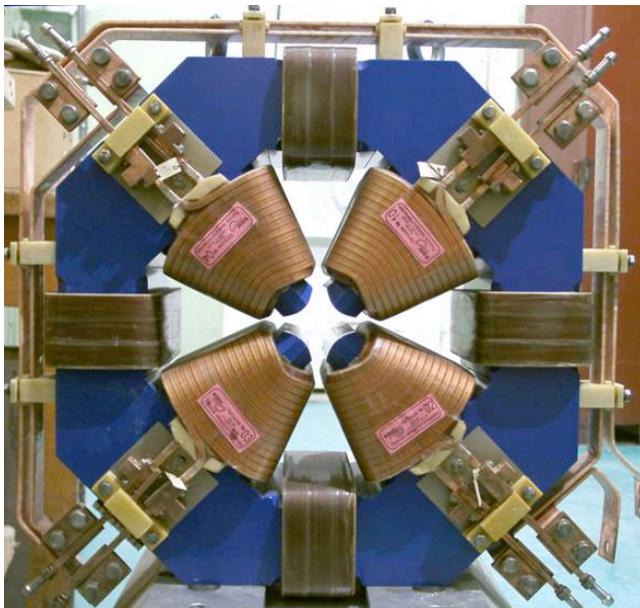
4. If the magnet has a symmetry, for example, one side is a mirror image of the other side, then the field lines will have that symmetry too.

5. The denser the number of field lines in a region, the stronger the magnetic field.

In actual practice, for example at TRIUMF, a beam line will have a continuous series of quadrupole magnets to keep the charged particle beam (like protons) within the beam line:

- one will focus vertically (for example) and defocus horizontally
- before the horizontal dimension of the beam gets too big for the pipe, the beam passes through another quadrupole that focuses horizontally and defocuses vertically
- eventually, before the vertical dimension gets too big for the pipe, the beam is again passed through a quadrupole that focuses vertically and defocuses horizontally
- the process repeats until the beam reaches its destination

A sequence of 6 quadrupole magnets around a beam line at TRIUMF .
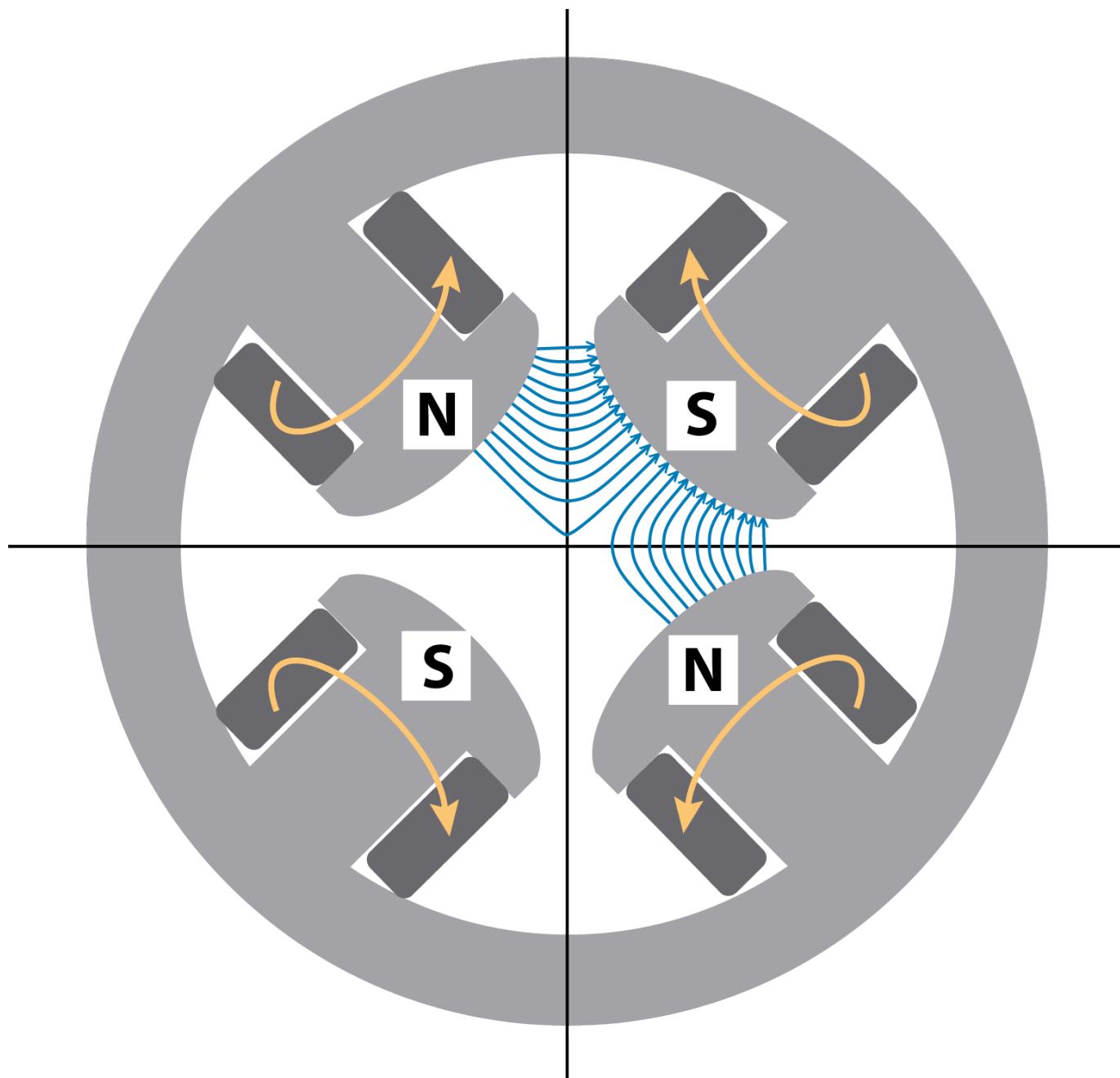
Head-on picture of a real quadrupole magnet. Note the curved shape of the pole tips, and the copper wire wrapping each pole to generate the magnetic field.

Worksheet 5.4 Magnetic Field Focusing, Quadrupoles

1) Field lines:
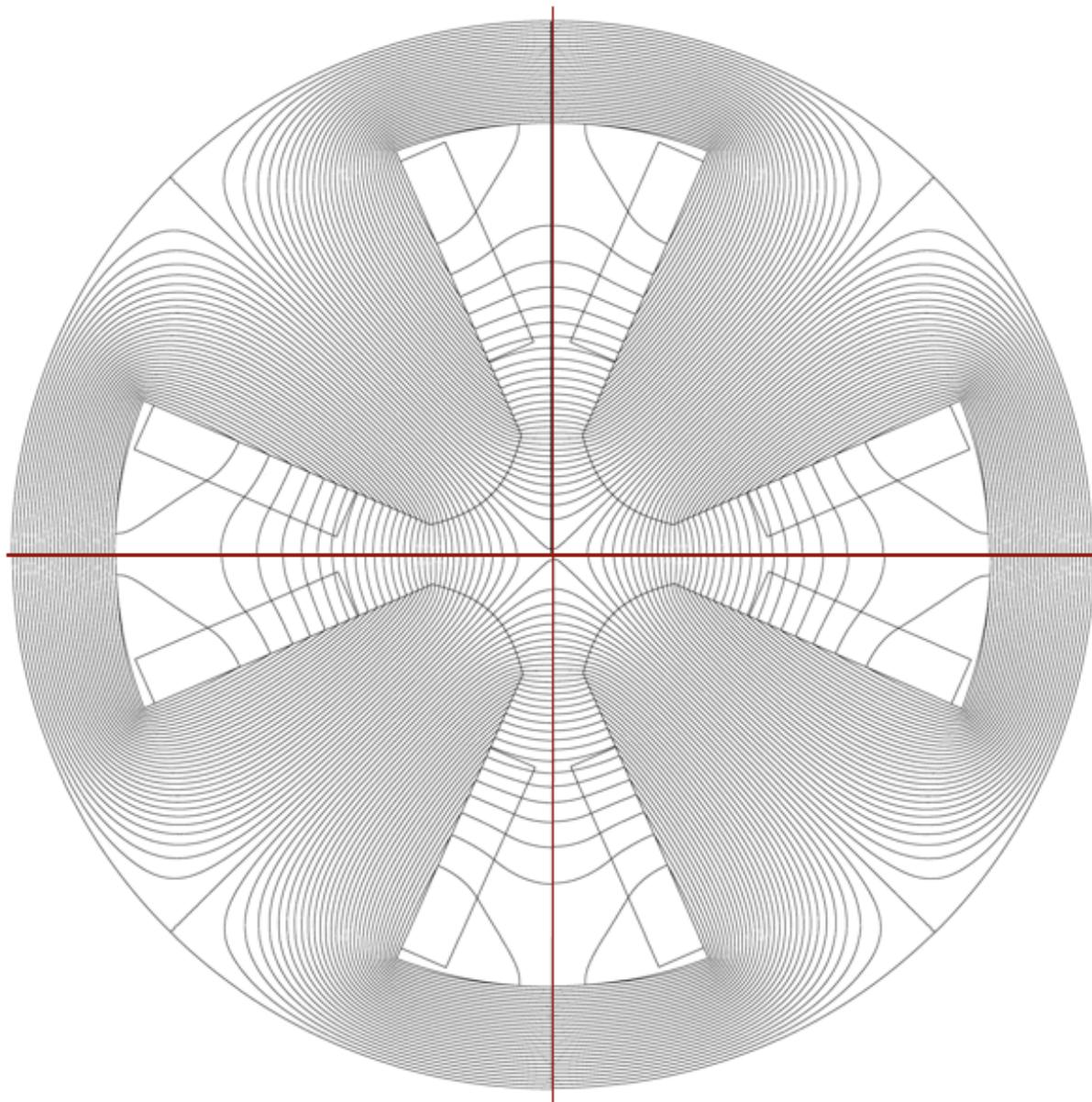
Refer to the figure below and try to draw some field lines for the cross-section of the quadrupole magnet shown. For clarity, starting points and end points are given to you.
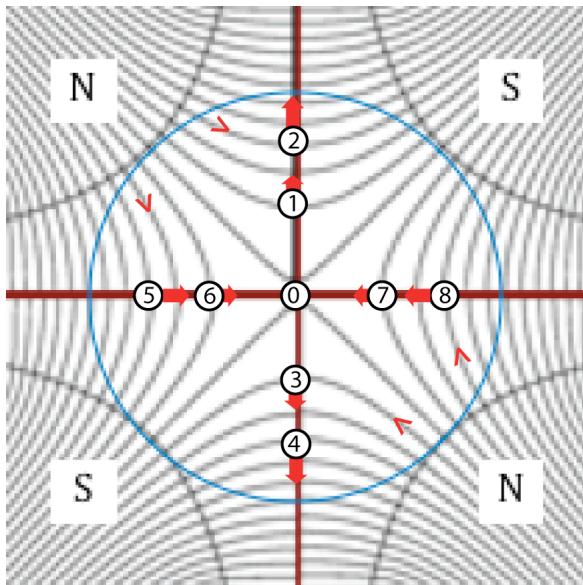
Drawn carefully, the magnetic field lines in the quadrupole magnet should look like those in the figure at right, which shows a detailed calculation of what the field lines look like, including inside the steel poles tips. The red lines show the symmetry axes of the magnetic field.

Note that no field lines cross the center, meaning there is no magnetic field at that point. Going outward from the center, one crosses field lines more and more frequently, meaning the magnetic field is getting progressively stronger. In fact, the curved shape of the pole tips ensures that the magnetic field gets proportionally stronger farther away from the axis, i.e. at twice the distance the field is twice as strong. This is just what we want from a focusing element.

In real life, the proportional relationship breaks down far away from the center near the narrow gap between pole tips. So designers must ensure that the beam size stays small.
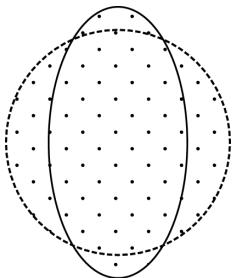
The picture to the left is a "close up" of the region in the quadrupole that the beam passes through, with field lines that are computer calculated:

2) Add arrows to show the direction of the field lines (from N to S).

3) Imagine a beam of protons passing through this field headed out of the page toward you. Using the right hand rule, determine the direction of the force on the protons at the labeled points. Draw an arrow showing the direction and SIZE of the force (remember, the further apart the lines are the weaker the field and the smaller the force). There are 8 points (1 to 8) and a zero point (where there is no effect, can you see why?)

4) Sketch what you think a circular beam would look like after going through the focusing magnet.

The important thing to note is that the magnetic forces cause the proton beam to *narrow* in one dimension, but *widen* in the other! So if it starts out with a circular cross section, it will be oval after it passes through the quadrupole. By simply drawing the field lines and analyzing the forces on our charged particle beam of protons, we have learned a basic feature of magnetic quadrupoles:

Quadrupole magnets focus in one dimension, but defocus in the other! Therefore, another quadrupole of the opposite polarity is required after the first one, to squeeze the beam in the opposite direction towards the original shape.

5) Explain in your own words why even though the beam is *defocused* in one direction when it is focused in the other, the result of going through two focusing elements (one focusing one way and the second in the other direction) results in a net focusing result in both directions.

Hint: consider how the sizes of the force on the particle are different in the first and the second quadrupoles.

Protons that are first focused then defocused begin far from the centre, so a large force pushes them inward. When they are defocused by the next magnet, they begin nearer the centre, so a smaller force pushes them out. The inward push is bigger in effect than the outward push. On the other hand protons that are first defocused then focused begin nearer the centre, so the outward force is small, then when they are focused they have reached a point further from the centre due to the defocusing, and the inward force is bigger. Thus in this case too the focusing effect is greater than the defocusing effect.

Lesson 6:   Beam Production and Filtering

6.1 Overview

Once the protons are extracted out of the cyclotron, they are put to use in a variety of ways. Here we focus on one application, namely using the proton beam to shatter large stable atoms into smaller exotic atoms used to study stars and supernovae. Let's follow what's covered in the video step by step.
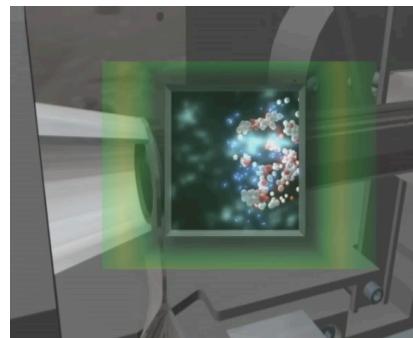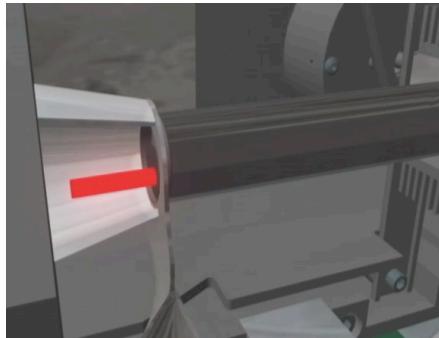
6.2 Creating and Filtering Exotic Isotopes by Their Mass

Step 1: Shatter a Large Nucleus to Create a Variety of Smaller Isotopes

The proton beam extracted from the cyclotron is guided and focused by magnets through an external beam line. At the end, it is directed at an *isotope production target*. Travelling at 75% the speed of light, the protons will completely shatter the target atoms into smaller fragments with a variety of neutrons and protons in them. Some fragments will be ordinary everyday atomic elements, but many will be exotic short-lived atoms not found on earth.

Tantalum is often used as a target material since it is a large nucleus (73 protons, 108 neutrons). It can break into a wide variety of fragments, and the metal has a high melting point (over 3000º Celsius) to withstand the energy deposited by the proton beam.

Targets and their cooling systems are designed to withstand the most intense proton beam that does not melt the material, since some of the most interesting exotic atoms are produced with very small rates. For example, scientists are very interested in the exotic lithium isotope lithium-11 (or ^{11}Li, with 3 protons and 8 neutrons), which is created at a rate of less than a hundred thousand per second (10^5) from a 80 microamperes proton beam (5×10^{14} protons per second!) hitting a tantalum target.

Step 2: Create and Ionize All Isotopes in the Beam

All the different types of isotopes created in the production target are collected as they exit the target into a beam pipe. At this stage they are all neutral atoms, so they must be ionized in order to be affected by electric and magnetic fields. However, unlike the hydrogen atoms for the cyclotron which have an extra electron added to become negatively charged, these isotopes are ionized by having an electron *removed*, resulting is a *positive* charge.

A number of techniques are used to remove an electron from the isotopes. For our purposes, all we need to know is that the end result is a beam mixture of different isotopes, by far most of which have had one electron removed (q =+1), a few with two (q = +2), much fewer with three (q = +3), and so on. **For the purpose of this lesson, let's assume that all the isotope ions have charge q = +1.** At the end of the lesson we'll discuss what happens with isotopes of larger charges (q = +2, +3, etc).
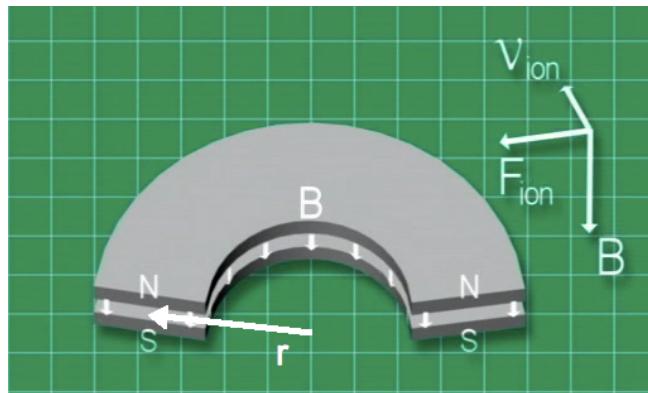
Step 3: Accelerate Isotopes in Beam to Same Kinetic Energy

Once we have our beam mixture of ionized isotopes, it's time to pick out the one particular isotope we want. Each isotope has a charge **(q = +1)** and a unique mass **(m)**. *No two isotopes have an identical mass* – they might be very nearly the same, making them difficult to distinguish, but their masses will always be different.

Step 4: Steer All Isotopes with Magnetic Field

As the isotope ion beam enters the dipole magnet, we know that each particular isotope has a velocity **v**, which depends on its mass **m**, its charge **q = +1**, and the electrical potential difference **ΔV**. This is just what we want, since we know that a magnetic field exerts a force on a charged particle proportional to its velocity. So each isotope ion will trace a *different* circular arc through the magnet, providing a way to separate them from each other. A magnet used in this way is called a *mass analyzer.*

The mass analyzer has a magnetic field **B**, perpendicular to the ions' direction of motion. By changing the field we can make different masses of ion bend at the radius the apparatus allows, so that one ion or another makes it through to our detectors and the others miss.
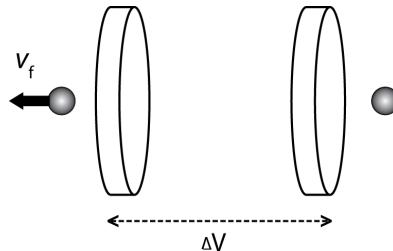
Step 5: Select Desired Exotic Isotope

We see that since the magnetic field **B** and the charge **q = +1** is the same for all the isotope ions, the mass analyzer magnet steers the beam in different circular arcs depending on velocity and mass. Since we know (and can control) the potential difference ΔV and the magnetic field **B**, and we are assuming that each isotope has a charge **q = +1** (to a good approximation), *the radius of curvature r depends only on the isotope mass m*. By placing a narrow slit (1 mm apart or narrower) at the exit of the mass analyzer magnet, only the isotope of mass **m** matching the magnetic field **B** will be able to pass through – all the others are blocked. So simply by varying the magnetic field, we can select different isotopes.

Worksheet 6.0   Preliminary Exercise: Beam Filtering

It may be easiest to start with a concrete example to help see what is going on in this process. We will start with a ^{8}Li atom (an important short-lived isotope) and a ^{16}O atom (note that the O is about twice the mass of the Li).

The video guides you in producing a more general expression.

Mass of ^{8}Li:	1.33×10^{-26} kg
Mass of ^{16}O:	2.66×10^{-26} kg
Single Charge:	1.60×10^{-19} C
Initial velocity:	≈ 0 m/s
ΔV:	40,800 V

Since we are interested in a general formula showing how these isotopes can be distinguished with electric and magnetic fields, let's work with the variables **m** and **q** and see what we get.

First, the beam is passed through an electric field created by an electrical potential difference **ΔV**.

1) What is the potential energy of each of these ions?

$E_P = q\Delta V = (1.60 \times 10^{-19}$ C$)(40,800$ J/C$) = 6.53 \times 10^{-15}$ J

(ans: 6.53×10^{-15} J)

2) This potential energy will become E_K as the atoms move across the potential difference. What is the kinetic energy of these atoms after they cross ΔV?

$E_K = E_P = 6.53 \times 10^{-15}$ J

(ans: 6.53×10^{-15} J)

3) What is the velocity of the ^{8}Li atom?

$E_K = \frac{1}{2}mv^2$ so $v = \sqrt{\dfrac{2E_K}{m}} = \sqrt{\dfrac{2(6.53x10^{-15}J)}{1.33x10^{-26}kg}} = 9.91 \times 10^5$ m/s

(ans: 9.91×10^5 m/s)
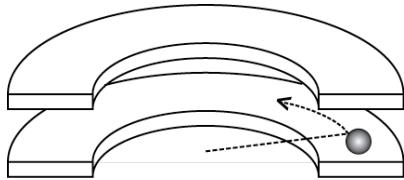
4) What is the velocity of the ^{16}O?

$E_K = \frac{1}{2}mv^2$ so $v = \sqrt{\dfrac{2E_K}{m}} = \sqrt{\dfrac{2(6.53x10^{-15}J)}{2.66x10^{-26}kg}} = 7.01 \times 10^5$ m/s

(ans: 7.01×10^5 m/s)

Notice that although the mass is twice as great, the velocity is not half as much, but there is a relationship. We will prove the relationship in the next worksheet.

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Now we are going to use a vertical magnetic field to bend the atoms along a curved path with a certain radius of curvature. For example the ^{8}Li is bent to a particular radius by a 1.266 T field:

Single Charge:	1.60×10^{-19} C
Mass of ^{8}Li:	1.33×10^{-26} kg
Velocity of ^{8}Li:	9.91×10^5 m/s
Magnetic Field:	0.258 T

5) What is the centripetal force on the ^{8}Li$^+$ ion?

$F_B = qvB = (1.60 \times 10^{-19}$ C$)(9.91 \times 10^5$ m/s$)(0.258$ T$) = 4.09 \times 10^{-14}$ N

(ans: 4.09×10^{-14} N)

6) What is the inward acceleration caused by this force?

$F_B = m\,a_c \qquad$ so $a_c = \dfrac{F}{m} = \dfrac{4.09 x 10^{-14} N}{1.33 x 10^{-26} kg} = 3.08 \times 10^{12}$ m/s^2
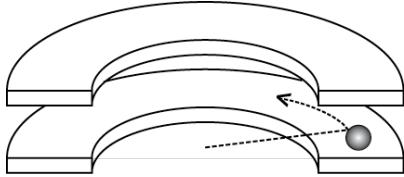
(ans: 3.08×10^{-12} m/s^2)

7) What is the radius of curvature given this acceleration?

$a_c = \dfrac{v^2}{r} \qquad$ so $r = \dfrac{v^2}{a_c} = \dfrac{(9.91 x 10^5 m/s)^2}{3.08 x 10^{12} m/s^2} = 0.319$ m

(ans: 0.319 m)

Now, we will find what magnetic field will give ^{16}O the same radius of curvature:

Single Charge:	1.60×10^{-19} C
Mass of ^{16}O:	2.66×10^{-26} kg
Velocity of ^{16}O:	7.11×10^5 m/s
Radius of curvature:	0.319 m

8) What inward force will give the ^{16}O the same radius of curvature?

$F_C = ma_c = m\dfrac{v^2}{r} = 2.66 x 10^{-26} kg \dfrac{(7.11 x 10^5 m/s)^2}{0.319 m} = 4.22 \times 10^{-14}$ N

(ans: 4.22×10^{-14} N)
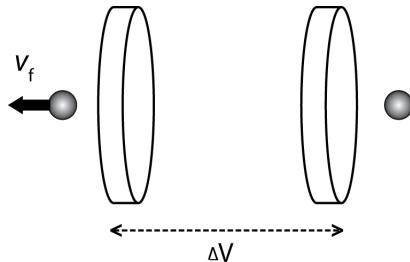
9) What magnetic field will produce that force on the ^{16}O$^+$ ion?

$F_C = qvB \qquad$ so $B = \dfrac{F}{qv} = \dfrac{4.22 x 10^{-14} N}{(1.60 x 10^{-19} C)(7.11 x 10^5 m/s)} = 0.371$ T

(ans: 0.371 T)

A larger field is needed, but again not twice as large. Next we'll derive a general expression to do such calculations in one step.

Worksheet 6.1 Beam Filtering

1) Write down the expression for the electrical potential energy E_P:

$E_P = q\Delta V$

2) Write down the expression for the kinetic energy E_K:

$E_K = \frac{1}{2} mv^2$

3) After the ions have passed through the electric field, the electrical potential difference will have been transformed into kinetic energy. Will the kinetic energy be the same, or different, for each type of isotope in the beam?

Same.

4) Assume the kinetic energy at the start was almost zero. By equating $E_K = E_P$, and rearranging the formula, determine an expression for the velocity v.

$E_K = E_P$

$\frac{1}{2} mv^2 = q\Delta V$

$v^2 = \dfrac{2q\Delta V}{m}$ $\qquad\qquad$ $v = \sqrt{2\Delta V \tfrac{q}{m}}$ $\qquad\qquad$ (ans: $v = \sqrt{2\Delta V \tfrac{q}{m}}$)

By passing all the isotope ions through the same electrical potential difference **ΔV**, we now know what each isotope's velocity **v** is from their mass **m**, and charge **q = +1**. However, the various isotope ions are still all mixed up together, so we need a way to separate them out in order to be able to select the one we want.

5) Write down the expression for the magnetic force F_{mag} on the isotopes of mass m, charge q, and velocity v.

$F_{mag} = qvB$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $F_{mag} = qvB$

6) Write down the centripetal force F_{ion} felt by the isotopes ions as they trace a circular arc of radius r through the magnet in terms of the speed of the ion.

$F_{ion} = ma_c = m\dfrac{v^2}{r}$ $\qquad\qquad\qquad\qquad\qquad$ (ans: $F_{ion} = m\dfrac{v^2}{r}$)

7) Equating the centripetal force with the magnetic force, $F_{ion} = F_{mag}$, rearrange the formula to get an equation for the radius of curvature r, traced out by each isotope ion.

$$F_{ion} = F_{mag}$$

$$m\frac{v^2}{r} = qvB$$

$$r = \frac{mv}{qB}$$

(ans: $r = \frac{mv}{qB}$)

We see that since the magnetic field **B** and the charge **q = +1** is the same for all the isotope ions, the mass analyzer magnet steers the beam in different circular arcs depending on velocity and mass. But we already know the velocity for each mass m.

8) Substitute the expression for the isotope ion velocity v in terms of the accelerating potential difference ΔV (from 4) in place of the velocity v in the expression for the radius of curvature r (in 7):

$$r = \frac{mv}{qB} \qquad\qquad \text{but } v = \sqrt{2\Delta V \tfrac{q}{m}}$$

so $\quad r = \frac{m\sqrt{2\Delta V \tfrac{q}{m}}}{qB} \qquad\qquad r = \sqrt{\frac{2\Delta V}{B^2}\frac{m}{q}}$

(ans: $r = \sqrt{\frac{2\Delta V}{B^2}\frac{m}{q}}$)

From this formula, we see how a particular isotope ion can be selected. Since we know (and can control) the potential difference **ΔV** and the magnetic field **B**, and we are assuming that each isotope has a charge **q = +1** (to a good approximation), *the radius of curvature **r** depends only on the isotope mass **m***. By placing a narrow slit (1 mm apart or narrower) at the exit of the mass analyzer magnet, only one radius if allowed out, and therefore only the isotope of mass **m** "matching" the magnetic field **B** will be able to pass through – all the others are blocked. Simply by varying the magnetic field, we can select different isotopes.

9) Rewrite the expression to find m/q as a function of the fixed vales r and ΔV, and the variable we change: B

$$r = \sqrt{\frac{2\Delta V}{B^2}\frac{m}{q}} \qquad\qquad r^2 = \frac{2\Delta V}{B^2}\frac{m}{q} \qquad\qquad \frac{m}{q} = \frac{r^2 B^2}{2\Delta V}$$

(ans: $\frac{m}{q} = \frac{r^2 B^2}{2\Delta V}$)

6.2 Isotope Mass Spectrum
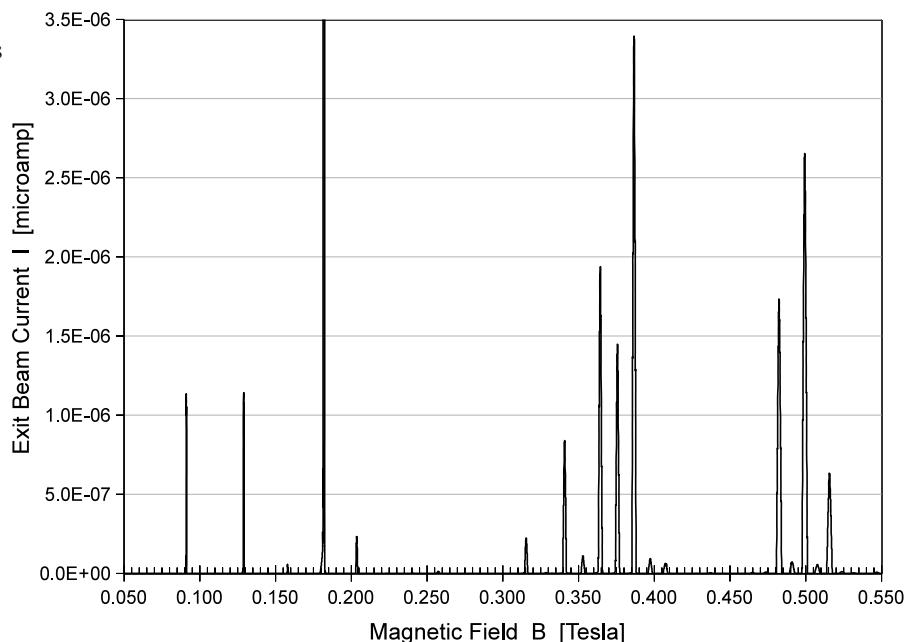
What Happens in the Real World

In practice, three things happen in real-life applications, which have not been considered in the idealized case:

1. The magnetic and electric fields are not perfect, and the slit width cannot be made infinitely narrow, so isotopes with masses very very close to the one we want *can* sneak though and be delivered by mistake to the experiments. Nonetheless, the technique is close to ideal and limits the "contaminating isotopes" to a small fraction of the total. It is up to the experiments using the beam to be able to distinguish the remaining few contaminating isotope ions.

2. Another problem occurs when isotope ions are *multiple-charged* (i.e. have more than one electron removed; charges $q = +2, +3$, etc.). Look again at the formula above – it depends on the fraction **m/q** not just **m**. So if the mass analyzer magnet is set up for an isotope of mass **m** with charge **q = +1**, another isotope in the beam with mass **2m** and charge **q = +2** has the same fraction **2m/2q = m/q**, so it too will be accepted by the analyzer! Same for isotopes with **3m/3q**, **4m/4q**, etc. Fortunately, in most cases the ionization process will create very very few multiple-charged isotope ions, so the contaminating isotopes can usually be handled by the experiments themselves. However, in some cases the contamination is too large, and so special ionizing techniques that are more selective about which isotopes are ionized, like laser ionization, must be used.

3. A third problem that arises is that the mixture of isotopes form *molecule*s before they enter the analyzing magnet. Or, molecules are present in the system (walls of pipes, etc.) and contaminate the isotope beam. Water (H_2O) is a common contaminant molecule. Sometimes the molecule's mass is different than any of the isotopes, making the molecules easy to spot, but other times their masses are very close, making it difficult or impossible to distinguish the molecule from an isotope. In those cases, scientists try to design their experiments to be able to distinguish one from the other.

The figure at right shows the number of isotopes exiting a mass analyzer magnet as its magnetic field is varied. You see peaks where an isotope's mass "matched" the magnetic field and the isotope got through, and places where the field did not bend any isotope the proper amount, so none got through.

This is part of an actual mass spectrum from TRIUMF, where only stable isotopes were created. There are no exotic isotopes in this spectrum, which will make it easier to analyze. At this stage, we still do not know which peak corresponds to which isotope. To determine that, we must convert the magnetic field along the x-axis into the corresponding isotope mass.

Atomic Charge and Mass units

When using the above formula, the result **m/q** will be in units of kilogram/Coulomb (kg/C), and from that it would be difficult to identify the isotope since neither the periodic table of the elements nor other sources tabulate the isotopes in that way. So we must convert m/q into units that scientists actually use.

For charge, scientists use *multiples* of the elementary charge

$$1e = 1.602 \times 10^{-19} \text{ Coulombs}$$

For example, if an ion isotope is missing 1 electron so its charge is positive $+1.602 \times 10^{-19}$ C, instead we write its charge as

$$q = +1.6022 \times 10^{-19} \text{ C} / (1.602 \times 10^{-19} \text{ C/e})$$

$$= +1e$$

or simply q = +1 for short. Similarly q = -1 if the ion isotope has an extra electron. Since all charge must be an integer multiple of 1e, scientists label electric charge as q = ±1, ±2, ±3, etc.

Scientists express atomic and isotope masses in terms of the *Atomic Mass Unit* (or A), which is how atomic and isotope masses are tabulated in the periodic table. Basically:

$$A = \textbf{Number of protons + Number of neutrons}$$

in an atom or isotope. For example, normal carbon has 6 protons and 6 neutrons, so we label it Carbon-12 (or ^{12}C) with mass A = 12.   Normal lithium has 3 protons and 4 neutrons, so we label it Lithium-7 or ^{7}Li with mass A = 7. To convert from atomic mass units to kilograms, multiply by:

$$1u = 1.6605 \times 10^{-27} \text{ kg/A}$$

where   **1u** *is defined to be* **= mass of ^{12}C/12**

So carbon-12 has a mass 1.9926×10^{-27} kg.

Conversely, if you know the isotope's mass **m**, then *divide* by 1u to convert it into an atomic mass A. Then simply look up that A value in the periodic table to identify the element.

Note that in the periodic table, most atoms show an atomic mass that is not exactly an integer, since the table shows the average atomic mass of all the stable isotopes. For example, carbon has two stable isotopes, roughly 99% of which is Carbon-12, and about 1% Carbon-13, which has 6 protons and 7 neutrons. So the periodic table shows carbon with mass A = 12.01, which is the average (0.99*12 + 0.01*13). For simplicity, a table of stable isotopes of all the atoms up to iron is included in this booklet.
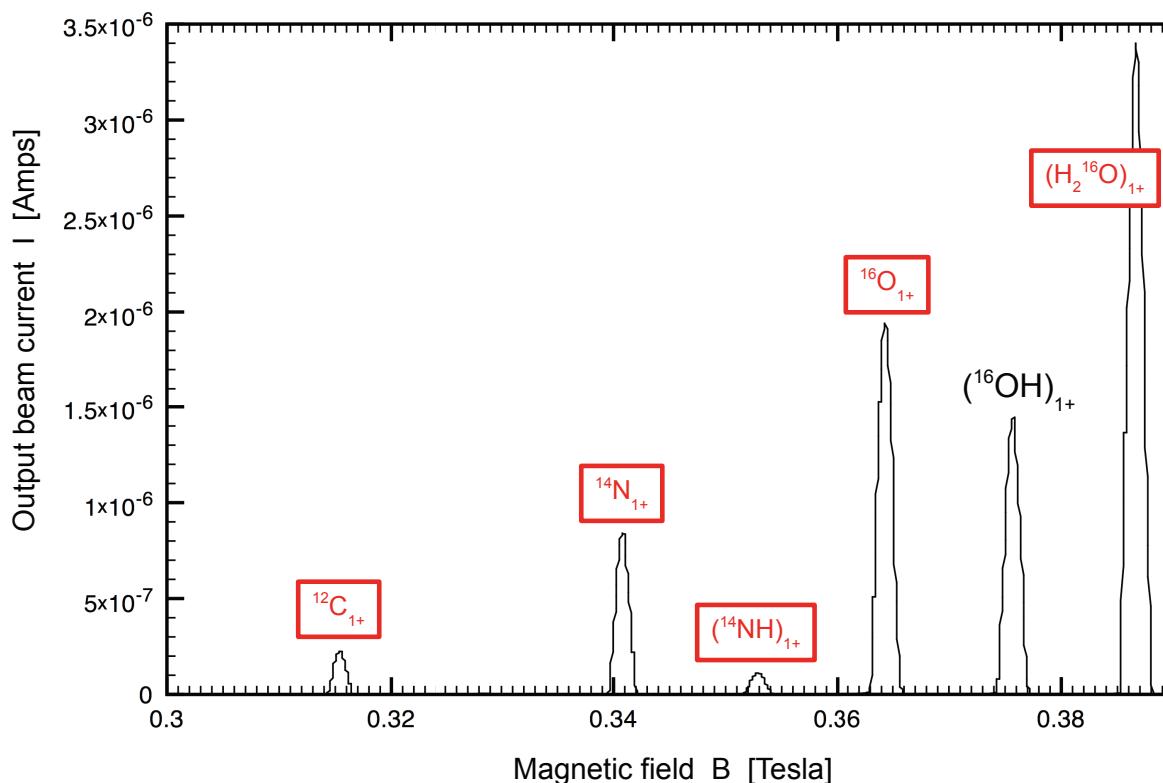
To summarize, when you calculate the fraction m/q in units of kg/C, you must multiply by

$$1 \text{ [A/e]} = 1 \text{ [kg/C]} * 1.6022 \times 10^{-19} \text{ [C/e]} / 1.6605 \times 10^{-27} \text{ [A/kg]}$$

$$1 \text{ [A/e]} = 1 \text{ [kg/C]} * 9.6489 \times 10^{7}$$

to convert it into units of A/e as scientists do. We will practice this on a real mass spectrum from TRIUMF in the next section.

Worksheet 6.2   Identify Isotopes in Stable Isotope Mass Spectrum from TRIUMF

Above is a part of an actual isotope spectrum from TRIUMF. The peaks represent the beam current at the exit of an analyzing magnet with radius of curvature r = 0.3195 m as the magnetic field is varied from 0.30 to 0.39 Tesla.
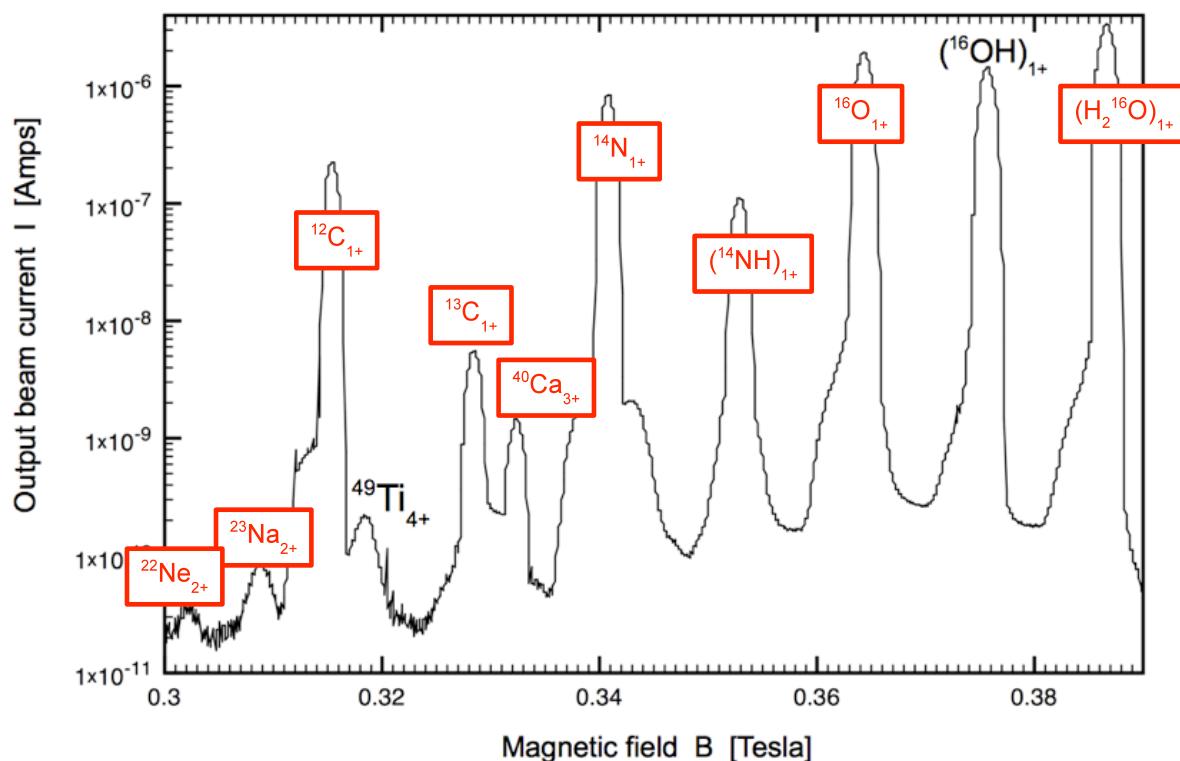
In the spectrum, there are 3 stable isotopes, and 3 stable *molecules*, all accelerated through a 40,800 Volt potential difference prior to entering the analyzing magnet. The single-charged OH radical molecule is identified for you

In the table below, calculate the m/q ratio for each of the peaks, using the B field determined from the graph and the parameters provided. Using the table of isotopes, see if you can identify each of the molecules and isotopes.

r = 0.3195 m   V = 40800 Volts

B [Tesla]	m/q [kg/C]	m/q [A]	Isotope/Molecule
0.3756	1.765×10^{-7}	17.03 (~17)	$(^{16}OH)_{1+}$

Worksheet 6.3 Challenging: Closer Look at Sample Isotope Mass Spectrum

Identifying the smaller peaks is more challenging: the spectrum has 3 molecules, 4 single-charged isotopes, and 4 multiple-charged isotopes. Two peaks are identified for you – one molecule on the right, and one multiple-charged isotope on the left. In the table on the next page, calculate the m/q ratio for each of the peaks, using the B field determined from the graph and the parameters provided. Using the table of isotopes, see if you can identify each peak.

r = 0.3195 m   V = 40800 Volts

B [Tesla]	m/q [kg/C]	m/q [A]	Isotope/Molecule
0.3756	1.765×10^{-7}	17.03 (~17)	$(^{16}OH)_{1+}$
0.3186	1.270×10^{-7}	12.25	$^{47}Ti_{4+}$

Appendix: Table of Stable Isotopic Masses and Natural Abundances

$1u=1.661 \times 10^{-27}$ kg

Z	Name	Symbol	Mass of Atom (u)	% Abundance
1	Hydrogen	^{1}H	1.007825	99.9885
		^{2}H	2.014102	0.115
2	Helium	^{3}He	3.016029	0.000137
		^{4}He	4.002603	99.999863
3	Lithium	^{6}Li	6.015122	7.59
		^{7}Li	7.016004	92.41
4	Beryllium	^{9}Be	9.012182	100
5	Boron	^{10}B	10.012937	19.9
		^{11}B	11.009305	80.1
6	Carbon	^{12}C	12.000000	98.93
		^{13}C	13.003355	1.07
7	Nitrogen	^{14}N	14.003074	99.632
		^{15}N	15.000109	0.368
8	Oxygen	^{16}O	15.994915	99.757
		^{17}O	16.999132	0.038
		^{18}O	17.999160	0.205
9	Fluorine	^{19}F	18.998403	100
10	Neon	^{20}Ne	19.992440	90.48
		^{21}Ne	20.993847	0.27
		^{22}Ne	21.991386	9.25
11	Sodium	^{23}Na	22.989770	100
12	Magnesium	^{24}Mg	23.985042	78.99
		^{25}Mg	24.985837	10.00
		^{26}Mg	25.982593	11.01
13	Aluminum	^{27}Al	26.981538	100
14	Silicon	^{28}Si	27.976927	92.2297
		^{29}Si	28.976495	4.6832
		^{30}Si	29.973770	3.0872
15	Phosphorus	^{31}P	30.973762	100
16	Sulphur	^{32}S	31.972071	94.93
		^{33}S	32.971458	0.76
		^{34}S	33.967867	4.29
		^{36}S	35.967081	0.02

Z	Name	Symbol	Mass of Atom (u)	% Abundance
17	Chlorine	^{35}Cl	34.968853	75.78
		^{37}Cl	36.965903	24.22
18	Argon	^{36}Ar	35.967546	0.3365
		^{38}Ar	37.962732	0.0632
		^{40}Ar	39.962383	99.6003
19	Potassium	^{39}K	38.963707	93.2581
		^{40}K	39.963999	0.0117
		^{41}K	40.961826	6.7302
20	Calcium	^{40}Ca	39.962591	96.941
		^{42}Ca	41.958618	0.647
		^{43}Ca	42.958767	0.135
		^{44}Ca	43.955481	2.086
		^{46}Ca	45.953693	0.004
		^{48}Ca	47.952534	0.187
21	Scandium	^{45}Sc	44.955910	100
22	Titanium	^{46}Ti	45.952629	8.25
		^{47}Ti	46.951764	7.44
		^{48}Ti	47.947947	73.72
		^{49}Ti	48.947871	5.41
		^{50}Ti	49.944792	5.18
23	Vanadium	^{50}V	49.947163	0.250
		^{51}V	50.943964	99.750
24	Chromium	^{50}Cr	49.946050	4.345
		^{52}Cr	51.940512	83.789
		^{53}Cr	52.940654	9.501
		^{54}Cr	53.938885	2.365
25	Manganese	^{55}Mn	54.938050	100
26	Iron	^{54}Fe	53.939615	5.845
		^{56}Fe	55.934942	91.754

Some notes about the solution:

- It would be natural to identify the A = 18 peak as Oxygen-18, but note its natural abundance is only 0.2%, whereas that peak is larger than the Oxygen-16 peak. In fact, water vapour from the air gets into the system during maintenance and gets embedded into the metal so that peak is almost all water (Hydrogen-1 and Oxygen-16), with a tiny bit of Oxygen-18 buried under the same peak. Water is a large problem in most mass spectrometers.

- The OH radical molecule comes from water, where one H was knocked off during the ionization process.

- Ammonia (NH) is present in lots of cleaning solutions, which can get into the metal, and is also formed in the ionization process as nitrogen (from air) happens to combine with hydrogen (from water).

- Also, it would be natural to identify the far-left peak with Boron-11, but in fact it is double-charged Neon-22. This particular ionization chamber once contained Neon-22, which apparently got absorbed into the metal, whereas Boron-11 is very very difficult to handle and has not been used in this apparatus.

- In the log plot, some peaks appear to have "shoulders". These are other isotopes or molecules which have nearby m/q values. For example, the quadruple-charged Titanium-48 peak lies underneath the Carbon-12 peak (both have same m/q) and the Titanium-47 peak is just to the left of the Carbon-12 peak, giving the latter a "shoulder". Natural titanium metal is used a lot for its relative lightness and strength, and obviously some of the atoms are getting into the apparatus. The shoulder to the right of Nitrogen-14 could be octuple-charged Indium-113, since indium is used for vacuum seals, and the one to the left is likely mostly quadruple-charged Manganese-55, since manganese is a popular element in metal alloys.