TRIUMF
Canada's national laboratory
for particle and nuclear physics
and accelerator-based science

2017 Student Seminar

Elementary Quantum Theory ($\frac{1}{4}$)

Presenter: Charlie Payne
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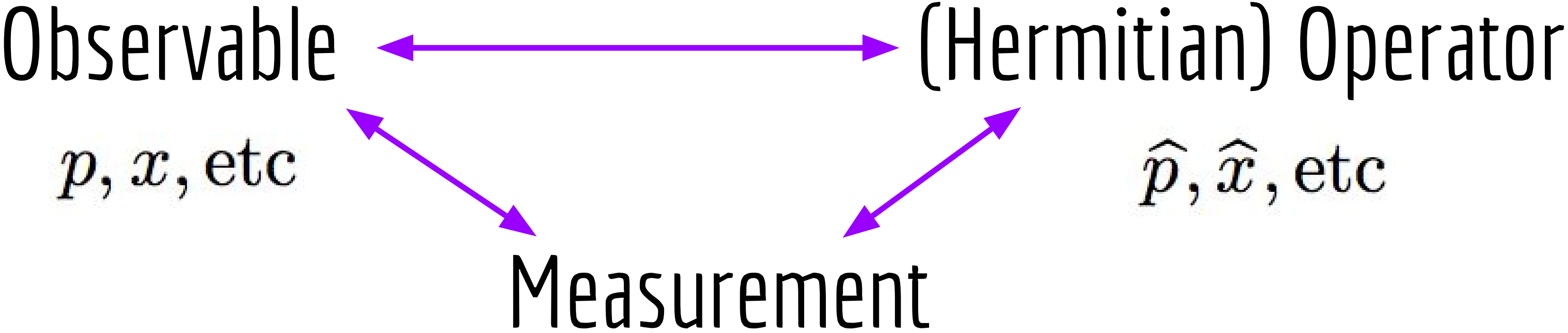
Outline

1) First Quantization Review
2) Commutators…
3) …and their Applications
4) Wave Packets
5) Time-Dependent Schrodinger Equation
6) Separation of Variables
7) Time-Independent S.E.


Circa: Winter 2011

(First) Quantization



Observable

p, x, etc

(Hermitian) Operator

$\hat{p}, \hat{x}, \text{etc}$

Measurement

$$\widehat{A} = \widehat{A}^{\dagger}, \quad \widehat{A}|\psi\rangle = a|\psi\rangle, \quad a \in \mathbb{R}$$

Spectral Theorem

Hermitian

$$\hat{A} = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i|$$

(Real) Eigenvalue

Eigenket

eg) $$|\lambda\rangle\langle\lambda| = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix} (c_1^*, c_2^*, \dots)$$

$$= \begin{bmatrix} |c_1|^2 & c_1 c_2^* & \cdots \\ c_2 c_1^* & |c_2|^2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

AKA: “Diagonalizable”

Math Speak

"Diagonalizable" =

$$\widehat{A} = U \Lambda U^\dagger, \quad UU^\dagger = \mathbb{1}$$

$$U = \left[\, |\lambda_1\rangle \, |\lambda_2\rangle \ldots \right]$$

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots)$$

Physics Speak

"Diagonalizable" =

We can find the eigenvalues and eigenvectors of an operator…

we'll leave the fancy organizing up to the mathematicians :D

Natural Units / Commutator

$$\hbar = c = \epsilon_0 = \mu_0 = 1$$

eg) $v = 0.8c \rightarrow 0.8$

$$s = \frac{\hbar}{2} \rightarrow \frac{1}{2}$$

$$[\widehat{A}, \widehat{B}] \doteq \widehat{A}\widehat{B} - \widehat{B}\widehat{A}$$

Commutator Application (1/3)

$$[\widehat{A}, \widehat{B}] = 0 \implies \widehat{A} \;\&\; \widehat{B}$$

are "simultaneously diagonalizable"
ie) they share the same eigenbasis

$$|\psi\rangle \to |a, b\rangle$$

Eigenvector

"Quantum Numbers"

$$\widehat{A}|a, b\rangle = a|a, b\rangle$$

$$\widehat{B}|a, b\rangle = b|a, b\rangle$$

Commutator Application (2/3)

$$\Delta A \Delta B \geq \frac{1}{2}\left|\langle[\widehat{A}, \widehat{B}]\rangle\right|$$

“Uncertainty Principle”

$$\text{where,}\quad \Delta A \doteq \sqrt{\langle \widehat{A}^2\rangle - \langle \widehat{A}\,\rangle^2}$$

“Standard Deviation”

$$\langle \widehat{A}\,\rangle \doteq \langle\psi|\widehat{A}\,|\psi\rangle = \int d\tau\,\psi^*(\vec{\boldsymbol{x}})\,\widehat{A}\psi(\vec{\boldsymbol{x}})$$

“Expectation Value”

Commutator Application (3/3)

Heisenberg EoM

$$\frac{d}{dt}\langle\widehat{A}\rangle = i\langle[\widehat{H},\widehat{A}]\rangle + \left\langle\cancelto{0}{\frac{\partial\widehat{A}}{\partial t}}\right\rangle$$

Hamiltonian

$$\text{ie)}\quad [\widehat{H},\widehat{A}] = 0 \implies \frac{d}{dt}\langle\widehat{A}\rangle = 0$$

Quantum Numbers

$$\implies a \text{ are constants of motion!}$$

Wave Packets

(diffraction experiments) → Matter ↔ Waves

(de Broglie)

$$\Psi(\vec{\boldsymbol{x}}, t) \propto \exp[i(\vec{\boldsymbol{k}} \circ \vec{\boldsymbol{x}} - \omega t)]$$

$$\vec{\boldsymbol{p}} = \hbar \vec{\boldsymbol{k}}$$

$$\Psi(\vec{\boldsymbol{x}}, t) = N \exp[\frac{i}{\hbar}(\vec{\boldsymbol{p}} \circ \vec{\boldsymbol{x}} - Et)]$$

$$E = \hbar\omega$$

(Einstein/Planck)

Wave Packets (cont'd)

(on board)

$$\vec{\boldsymbol{p}} \longrightarrow \hat{\boldsymbol{p}} = -i\vec{\boldsymbol{\nabla}}$$

$$\vec{\boldsymbol{p}}\Psi = -i\vec{\boldsymbol{\nabla}}\Psi$$

likewise,

$$E \longrightarrow \widehat{E} = i\frac{\partial}{\partial t}$$

Time-Dependent Schrodinger Equation

(Operator) $H = \frac{p^2}{2m} + V \quad \longleftrightarrow \quad E$ (Eigenvalue)

$$\boxed{\widehat{H}\Psi(\vec{x},t) = i\frac{\partial}{\partial t}\Psi(\vec{x},t)}$$

(NR)-TDSE

1D-TDSE

$$\widehat{\boldsymbol{p}} = -i\vec{\boldsymbol{\nabla}} \quad \Longrightarrow \quad \frac{\widehat{\boldsymbol{p}}^2}{2m} = -\frac{1}{2m}\nabla^2 \quad \Longrightarrow \quad \widehat{H} = -\frac{1}{2m}\nabla^2 + V(\vec{\boldsymbol{x}})$$

$$\boxed{\text{eg)} \quad i\frac{\partial}{\partial t}\Psi(x,t) = -\frac{1}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x)\Psi(x,t)}$$

(one-dimensional)

Separation of Variables

Guess: $$\Psi(x,t) = \psi(x)T(t) \mapsto \text{S.E.}$$

$$\frac{1}{\psi T}\left(i\psi\frac{\partial T}{\partial t} = -\frac{T}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi T\right)$$

LHS:
time-dep.

$$i\frac{1}{T}\frac{dT}{dt} = -\frac{1}{2m\psi}\frac{d^2\psi}{dx^2} + V(x)$$

RHS:
space-dep.

Must equal a constant!

Separation of Variables (cont'd)

…LHS)

$$\Longrightarrow \quad i\frac{1}{T}\frac{dT}{dt} \equiv E_n$$

"separation <u>constant</u>"

$$\therefore \; \frac{dT(t)}{dt} = -iE_nT(t) \quad \Longrightarrow \quad T(t) = e^{-iE_nt}$$

World's blurriest "therefore"

World's easiest ODE

Time-Independent Schrodinger Equation

...RHS)

$$-\frac{1}{2m}\frac{d^2\psi_n}{dx^2}+V(x)\psi_n=E_n\psi_n$$

$$\boxed{\widehat{H}\psi_n=E_n\psi_n}$$

TISE

general solution to the 1D-TDSE

$$\Psi(x,t)=\sum_n c_n\psi_n(x)e^{-iE_nt}$$

ODE solutions to the TISE

Thanks!

About to J-walk to TRIUMF,
Google caught me...

Questions?