Derivation and Application of Pules shapes in Radiation Detectors: Idiot’s Guide to Ramo’s Theorem

Greg Hackman
Outline

or what I hope you will get out of this

• Signals from radiation-induced events on electronic detectors evolve with time while charge carriers are moving

• Time dependence of signal depends on charge carrier trajectory and configuration of conductors

• Part of calculating the signal shape requires a theoretical **weighting field** calculated with the electrode of interest at a potential of 1 and the remaining conductors set to ground

• Once the principle is established, simple arguments using weighting fields are often adequate to explain detector signals and to design better detectors.
Generic electronic radiation detector

- Ionizing radiation interacting with matter liberates positive and negative charge carriers
- **Negative**: electrons
- **Positive**: ions (atomic or molecular) in gas, holes in semiconductors
- This example: radiation event causes very localized energy transfer to detection medium
  - Examples: neutral particle interaction (gamma, neutron, neutrino, dark matter), or in-medium beta decay
- generating a single pair of charge carriers, net charge zero.
Generic electronic radiation detector

- Applied electric field makes charge drift
- **Electrons** are generally fast; **holes**, slower; **ions**, very slow
- Current-sensitive (proportional or integrating) circuits measure signal
- Total charge (integrated current) is proportional to energy transferred by radiation to detection medium
Generic electronic radiation detector

HOWEVER: you don’t just get an instant blip when the charge carrier hits

• Charge on contact is induced while the charge carrier moves

• That is, the meters outside the detector measure a time-varying current as long as the charge carriers are moving
Example: Real signals from a coaxial detector

- Bulleted HPGe detector
- Big single crystal of germanium about the size of your fist
- Hole drilled not quite all the way through the middle
- Electrical contacts formed by heavy doping on core and on outside
- Typical for high energy-resolution gamma-ray spectroscopy
Real signals from a coaxial detector

- Operated as a reverse biased diode
- Voltage supply typically 1000 to 5000 V
- Signals HV decoupled and pre-amplified
Real signals from a coaxial detector

- Real detector, $^{60}$Co source,
- Preamplifier with exponential response, 50 ns 10%-90% rise time

- Rise time of this signal, more like 170 ns
- What gives?
Real signals from a coaxial detector

- top-left ("Convex") signals actually rare,
- right signals ("Concave") more common
- other pulse shapes like bottom left also seen
• Why do they look the way they do?

• First some electrostatics review ...
A very quick review of electrostatics

• Gauss’ law for electric field from charge distribution in absence of polarizable media:

\[ \iint \vec{E} \cdot \hat{n} \, dA = \frac{Q}{\varepsilon_0} \quad \text{or} \quad \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \]

• Electric potential:

\[ \nabla \cdot V = -\vec{E} \]

• Double-Differential:

\[ \nabla^2 V = -\frac{\rho}{\varepsilon_0} \]
A quick review of conductors

- We will treat our contacts as finite conductors and our charge carriers as infinitesimally small conductors.
- On an ideal conductor, any net charge is on its surface.
- The potential throughout a conductor, including its surface, is constant.

- Electric field $E$ at surface of conductor is perpendicular to surface and is proportional to surface charge density, $E = \frac{\sigma}{\epsilon}$
  - But E-field is NOT necessarily constant, so neither is surface charge density.
- All this is true whether the charge is on the conductor because you (or whoever) put it on there as a fixed value $Q$, with $V$ as a consequence; or whether it naturally flowed to the conductor on application of a fixed EMF $\mathcal{E}$, with $Q$ as a consequence.
Start with a trivial example:
Induced charge on a an infinite grounded conductor from a nearby free charge

• Boundary condition:
  Voltage at grounded conductor is 0

We’ll make the charge q move in a bit. Be patient.
Use method of mirrors

- Boundary condition: Voltage at grounded conductor is 0 at planar surface of conductor

- This $V=0$ along this plane is also a solution if the conductor is replaced with a negative charge on the other side of the plane – the mirror charge.

$$E_{\perp} \propto q \frac{Z}{(\sqrt{z^2 + r^2})^3}$$
Surface charge distribution is needed to give perpendicular field. Local charge magnitude varies in strength. Opposite sign to that of free charge $q$. (Size of symbols roughly proportional to magnitude.)
Let’s break infinite solid conductor into three segments, middle one finite, but connected by ideal conductor at surface.
Net charge on segments is just integral of surface charge
Move charge closer to conductor: fields at centreline get larger, at edges get smaller: Centre conductor charge \(-Q\) has a larger magnitude
Moving a POSITIVE free charge TOWARDS the centre conductor results in a net flow of NEGATIVE charge INTO the conductor – which equals a net POSITIVE flow OUT OF the conductor.

(keeping track of signs is the hardest part of this for me)

\[ V = 0 \]

\[ \Delta Q \text{ OUT of centre same sign as } q \]
Connect left and right together to the – side of a current-sensitive scope, centre contact to + side (behaves like a perfect conductor inside so $\Delta V=0$ across scope)

$$V=0$$

$$i(t) = \frac{dQ}{dt}$$
Initial charge $q$ at position $r(t)$, net charge on centre conductor $Q(t) = -Q$, and $dQ/dt = 0$.
Move charge $q$ to position $r(t+\Delta t) = r(t) + \Delta r$
Charge on centre contact changes from $-Q$ to $-Q - \Delta Q$
$\Delta Q$ flows out of centre during $\Delta t$
Take-away point: As viewed externally, current flows out of the centre contact as the charge moves towards it. (There is NOT a sudden delta function of current when the charge hits the conductor)
You can also integrate the current to get a total charge. Total charge will be proportional to energy loss in radiation event, independent of $\Delta t$. 
This happens for charges drifting but NOT hitting the contact too.
This happens for charges drifting but NOT hitting the contact too
This happens for charges drifting but NOT hitting the contact too:

Current signal

\[ I = \frac{\sigma}{\epsilon} \]
This happens for charges drifting but NOT hitting the contact too:
Charge signal (time-integrated current)
Consequences for a real detector:

- Arrow indicates POSITIVE current flow
- Color of arrow indicates which charge is inducing it
- Negative electron moving towards anode increases positive charge on anode
- There needs to be positive current flow through $E_A$ INTO the anode to increase the charge.
Consequences for a real detector:

- Positive Hole moving towards cathode induces positive flow out
  - Cathode gets more negatively charged
  - Negative flow into cathode through $\mathcal{E}_c$
  - Negative flow in equals positive flow out.
Consequences for a real detector:

- BUT
- Hole moving AWAY from ANODE also induces current on anode!
  - Anode gets less negative i.e. more positive!
  - Net positive flow into anode due to hole moving away from it!
Consequences for a real detector:

- AND
- Electron moving AWAY from CATHODE induces charge on cathode while it’s moving!
  - Cathode gets less positive
  - Net positive current OUT of Cathode due to electron moving AWAY from it!
Method of mirrors falls apart quickly, though

• I neglected the mirror charge induced on the anode due to the induced charge on the cathode
• And vice versa
• Infinite series
• Impractical to solve quantitatively for complicated geometries

• So how do we solve the current from the anode $\frac{dQ_A}{dt}$ as the electrodes drift towards it and the holes drift away from it?
A quick notational point

• You will see this notation a lot.
• V is some quantity which is (probably) dependent on position.
• Right subscript j refers to a specific object – a contact, a charge carrier, whatever.
• Left superscript X refers to a configuration of boundary conditions – fixed charges and EMFs.
• \( A \, Q \, C \) means the charge (Q) on the cathode (C) when voltage is applied to the anode (A).
Superposition principle in a nutshell:

• The consequences of an assembly of charges and EMF sources is the same as the linear sum of the consequences of each individual charge and source
The electric field that causes the drift of the charge carriers due to EMF applied to the contacts labeled anode and cathode, $E(r)$, can be calculated for two configurations: $^A E(r)$ with the anode held at EMF and the cathode power turned to zero volts (i.e. connected to ground), and $^C E(r)$ with the cathode power turned on and the anode set to zero volts. (This is how SIMION works)
Superposition of applied EMF:

\[ Q = \text{Anode} + Q \]

Similarly the potential field and the charge distributions – in this case, the charge on each contact – with both sources on, is the sum of what you get with one on and the other off.

NOTE: \( Q \) can be nonzero even if EMF is zero!
Superposition of charged thingy:

- Add a free charge carrier as an infinitesimally tiny conductor with a fixed charge on it and no fixed potential
  - (but maybe we can measure it, with a voltmeter if we wanted to).
- Now the total charge on the anode is that arising from applied EMF, plus that arising from the fixed charge with the EMF sources set to 0.
- We will be using the leftmost configuration later ....
Superposition principle applied to a detector:

- Ultimately we want to know $dQ_A/dt$, the current on a contact.
- The EMF supplying the drift field is fixed and constant, so $d(\Delta_Q Q_A)/dt$ is zero.
- So to get the response due to radiation induced free charge carriers, ...
Our strategy:

- Figure out where \( q \) is as a function of time \( r_q(t) \) (easy)
- Calculate \( qQ_A \) for all these positions (?????)
- And apply a lazy calculus chain rule,
  \[
  I = \frac{d(qQ_A)}{dt} = \left( \frac{d(qQ_A)}{dr} \right) \cdot \left( \frac{dr}{dt} \right) \] (easy)
- This will give you the current signal from the drift of radiation-induced charge carriers.
Continuing on ...

For a given configuration $X$ of EMFs $^{X}\mathcal{E}_i$ and fixed charges $^{X}q_i$ on our ideal detector:

- $^{X}V(r)$ is fixed at conductor surfaces by the applied EMF: $^{X}V(r)=^{X}V_i=^{X}\mathcal{E}_i$ for $r$ on the conductor surface
- Charge density $^{X}\rho(r)$ will be non-zero only on the surfaces of conductors
- Each conductor will have a total charge $^{X}Q_i$, either set explicitly as a fixed charge $^{X}q_i$ or as a consequence of applied EMF $^{X}\mathcal{E}_i$
Mixing up some math & physics

• Let \((-1/\varepsilon)\) be constant (homogeneous medium)
• If you were to take a volume integral of the quantity \((XV(r))(\nabla^2 XV(r))\)
  around a conductor (don’t ask why, yet)

\[
\iiint_{\mathcal{A}} \left( XV(\vec{r}) \cdot \nabla^2 XV(\vec{r}) \right) dr^3 = ?
\]
More math

\[ \iiint_{\mathcal{A}} \left( X V(\vec{r}) \cdot \nabla^2 X V(\vec{r}) \right) d\mathbf{r}^3 = \ ? \]

- \( \nabla^2 X V(\vec{r}) \) is proportional to \( \rho(\vec{r}) \), the charge density
- All the charge is on the surface of the conductor – it’s zero everywhere else
- So the volume integral reduces to a surface integral on the surface of the conductor

\[ \iiint_{\mathcal{A}} \left( X V(\vec{r}) \cdot \nabla^2 X V(\vec{r}) \right) d\mathbf{r}^3 = \iint_{\mathcal{A}} \left( X V(\vec{r}) \cdot \left( \frac{-1}{\varepsilon_0} X \sigma(\vec{r}) \right) \right) dA \]
More math

- The voltage is constant on the entire surface by definition of a conductor...

\[
\iiint (\vec{V}(\vec{r}) \cdot \frac{x}{\varepsilon_0} \sigma(\vec{r})) \, dA = \frac{-1}{\varepsilon_0} x V_A \cdot \iiint x \sigma(\vec{r}) \, dA
\]

- And the integral of the charge density is just the total charge on the conductor

\[
\frac{-1}{\varepsilon_0} x V_A \cdot \iiint x \sigma(\vec{r}) \, dA = \frac{-1}{\varepsilon_0} x V_A \cdot x Q_A
\]
More math

\[ \iiint_{\text{big volume}} \left( X V(\vec{r}) \cdot \nabla^2 X V(\vec{r}) \right) \, dr^3 = \frac{-1}{\varepsilon} X V_A X Q_A \]

- If you now take this integral over a large volume containing a collection of conductors:

\[ \iiint_{\text{big volume}} \left( X V(\vec{r}) \cdot \nabla^2 X V(\vec{r}) \right) \, dr^3 = \frac{-1}{\varepsilon} \sum_{j=1}^{\text{all of them}} X Q_j X V_j \]

- (Since Q and V are ordinary scalar numbers I can reverse them as factors in the terms.)

- This equation in itself is not useful but the derivation will be .....
Green’s Identity

• The next step is to apply one of the Green’s Identities.

• Keep the divergence theorem in mind: If \( \mathbf{A}(\mathbf{r}) \) is a vector field, the “flow” of this field across a closed surface (the integral of components perpendicular to the surface) is equal to the volume integral of the divergence of \( \mathbf{A}(\mathbf{r}) \) within the full volume enclosed by the surface

\[
\oint_S (\mathbf{\tilde{A}} \cdot \hat{n}_S) \, dS = \iiint_V (\nabla \cdot \mathbf{\tilde{A}}) \, dV
\]
Green’s Identity

• The next step is to apply one of the Green’s Identities.

• If $F(\mathbf{r})$ and $G(\mathbf{r})$ are two scalar functions:

\[
\nabla \cdot (F(\mathbf{r}) \nabla G(\mathbf{r})) = (\nabla F(\mathbf{r})) \cdot (\nabla G(\mathbf{r})) + F(\mathbf{r}) \nabla^2 G(\mathbf{r})
\]

and

\[
\nabla \cdot (G \nabla F) = (\nabla G) \cdot (\nabla F) + G \nabla^2 F
\]

• Note that the gradient dotted with the gradient is a scalar
Green’s Identity

\[ \nabla \cdot (F(\vec{r}) \nabla G(\vec{r})) = (\nabla F(\vec{r})) \cdot (\nabla G(\vec{r})) + F(\vec{r}) \nabla^2 G(\vec{r}) \]

and

\[ \nabla \cdot (G \nabla F) = (\nabla G) \cdot (\nabla F) + G \nabla^2 F \]

Subtract and Integrate

\[ \iiint_V \nabla \cdot (F \nabla G - G \nabla F) dV = \iiint_V \left( F \nabla^2 G - G \nabla^2 F \right) dV \]

Apply divergence theorem:

\[ \oiint (F \nabla G - G \nabla F) \cdot \hat{n} dS = \iiint \left( F \nabla^2 G - G \nabla^2 F \right) dV \]
Green’s Identity

• (or at least the one we need)
• If F and G are scalar fields on r,

\[
\iiint (F \cdot \nabla^2 G - G \cdot \nabla^2 F) \, dr^3 = \iint \left( F \frac{\partial G}{\partial n} - G \frac{\partial F}{\partial n} \right)
\]
Green’s Identity Special Case

• If the surface integral is taken where \( F \) and \( G \) are zero:

\[
\iiint (F \cdot \nabla^2 G - G \cdot \nabla^2 F) \, dr^3 = 0
\]

\[
\iiint (F \cdot \nabla^2 G) \, dr^3 = \iiint (G \cdot \nabla^2 F) \, dr^3
\]
Let’s add some physics

\[ \iiint (F \cdot \nabla^2 G - G \cdot \nabla^2 F) \, dr^3 = \iiint \left( F \frac{\partial G}{\partial n} - G \frac{\partial F}{\partial n} \right) \]

• Let the function \( F \) be a potential \( FV(\mathbf{r}) \) arising from a bunch of conductors with applied EMFs \( F\varepsilon_i \) and charges \( FQ_i \).

• Let the function \( G \) be a potential \( GV(\mathbf{r}) \) arising from the same conductors with different applied EMFs \( G\varepsilon_i \) and charges \( GQ_i \).

• And let’s take the both of them to go to zero at the boundary of the integral
  – E.g. a grounded outer housing of a detector
  – Or infinitely far away
Turn $F$ and $G$ into potentials $F_V$ and $G_V$

$$
\iiint F_V \cdot \nabla^2 G_V \, dr^3 = \iiint G_V \cdot \nabla^2 F_V \, dr^3
$$

Apply relationship between charge and Laplacian of potential

$$
-\frac{1}{\varepsilon} \iiint F_V \cdot G \rho \, dr^3 = -\frac{1}{\varepsilon} \iiint G_V \cdot F \rho \, dr^3
$$
Same “math by physics” again

\[ \iiint_{V} (F_{V}(\vec{r}) \cdot G \rho(\vec{r})) \, dr^3 = ? \]

- $G \rho$ is non-zero only on the conductor surface
- so this volume integral around the conductor becomes a surface integral on the surface of the conductor again
- On that surface $F_{V}(\vec{r}) = F_{V_{i}}$ and constant – factor out of integral
- The remaining integral of $G \rho$ is just the total charge $G Q_{i}$
Applying electrostatics of conductors and charges to Green’s Identity

\[ \iiint F V \cdot \nabla^2 G V \, dr^3 = \iiint G V \cdot \nabla^2 F V \, dr^3 \]

\[ \frac{-1}{\varepsilon} \iiint F V \cdot G \rho \, dr^3 = \frac{-1}{\varepsilon} \iiint G V \cdot F \rho \, dr^3 \]

\[ \frac{-1}{\varepsilon} \sum_{j=1,n} \iiint_j F V \cdot G \rho \, dr^3 = \frac{-1}{\varepsilon} \sum_{j=1,n} \iiint_j G V \cdot F \rho \, dr^3 \]

• Becomes this sum over conductors:

\[ \sum^F Q_i G V_i = \sum^G Q_i F V_i \]
What does this mean?

\[ \sum F Q_j G V_j = \sum G Q_j F V_j \]

- Say you have a configuration of conductors 1 \( \ldots \) \( j \)
- In one case you apply fixed voltages (including zero) or fixed charges (including zero) to (some of) them
- In the second case you apply a different set of fixed voltages (including zero) or fixed charge (including zero) to (some of) them
- Due to the mathematics of Green’s theorem and the physics of conductors, the above relationship must and will hold.
  - There isn’t an intuitive handwaving way to explain this \( \ldots \) it’s just a consequence of the math.
How do we use this?

\[ \sum F Q_j G V_j = \sum G Q_j F V_j \]

The strategy: Our conductors will be the components of our detector.

• One configuration of voltages and charges describes a free charge \( q \)
liberated by radiation within the detector.
• Invent a purely mathematical TEST configuration
• Design that test configuration so that when you put it in the
equation above, you get an equation for the charge \( Q_A \) on
conductor A (the anode) that depends on the free charge \( q \)
• Let the free charge move through the detector
• Calculate the change in anode charge with time – that’s our time-
dependent detector signal.
This is the configuration in which we want to find \( qQ_A \) as \( q \) at position \( r(t) \) drifts to the anode.

- Replace the charge carrier with a charged, infinitely small conductor
- The charge carrier is not connected to any potential source; so there’s no EMF source to turn off (unlike the anode & cathode)
- So our configuration has three conductors: \( A,C,q \)
Configuration $q$

- Three conductors: $A, C, q$
- $qQ_q = q$ is our drifting charge
- $qV_A = 0; qV_C = 0$
- $qV_q = \text{don’t care, it turns out}$
- $qQ_A$ is a consequence of our boundary conditions – i.e. the charge on $q$ -- and is what we want to calculate
Configuration TestA

Invent a configuration TestA

- Same collection of conductors: A, C, q
- Put a potential $W$ on the contact of interest, in this case, the anode A
- Set EMF power supplies to zero on everything else (in this case, keep the cathode grounded)
  - $TestA\phi_A = W$; $TestA\phi_C = 0$
- Put zero charge on charge carrier, $TestAQ_q = 0$
- Charge carrier conductor still unconnected to any EMF source, so it will be at $TestA\phi(r)$ at its location
We call \( \text{TestA} \phi(r) \) a test potential or a weighting potential.

Important note: this is a purely mathematical configuration! (that’s why I used \( \phi \) instead of \( V \)).

However, it does give us a function \( \text{TestA} \phi(r) \) that can go into Green’s Identity so we can figure out what’s going on in the real configuration.
For this case:

$$\sum (TestA Q_i)(qV_i) = \sum (qQ_i)(TestA \phi_i)$$

TestA $$\phi_q = TestA \phi (r)$$
Remember what we want to solve for:

\[ \text{TestA} \varphi_q = \text{TestA} \varphi (r) \]
Substitute known values from TestA config

\[ \text{TestA} V_A = W \]
\[ \text{TestA} V_C = 0 \]
\[ \text{TestA} Q_A = 0 \]
\[ \text{TestA} Q_C = 0 \]

\[ qQ_A (W) + qQ_C (0) + qQ_q \phi_q \]

\[ \text{TestA} \phi_q = \text{TestA} \phi (r) \]
Substitute known values from $q$ configuration:

$$\text{Test}_A Q_A(0) + \text{Test}_A Q_C(0) + (0)^q V_q = q Q_A(W) + q Q_C(0) + q \phi_q(r)$$
Getting there:

\[ qQ_A W_A + qQ_C (0) + (0) qV_q = qQ_A W_A + qQ_C (0) + q TestA \varphi_q \]

- Lots of zero factors reduce relevant terms

\[ 0 = qQ_A(W) + q TestA \varphi(r) \]

- Rearrange a bit

\[ qQ_A = -q \cdot TestA \varphi(r) \]
And here it is: Ramo’s Theorem.

\[ qQ_i(t) = -q \frac{\text{Test}_A \phi_i(r(t))}{W} \]
And here it is: Ramo’s Theorem.

\[ qQ_i(t) = -q \frac{\text{TestA} \phi_i(r(t))}{W} \]

In an assembly of conductors and a free charge:
The net charge induced on one of the conductors by the free charge is given by the value of a theoretical potential at the location of the free charge calculated when the conductor of interest has a unit potential applied and the remaining contacts are held at a common ground potential.
And here it is: Ramo’s Theorem.

\[ qQ_i(t) = - q \frac{\text{TestA} \varphi_i(r(t))}{W} \]

We derived it explicitly for the anode.

Generalized, it applies to any conductor in any arbitrary assembly of conductors.

The theorem is often written with \( W = 1 \), which we will from here on out.
How much charge is induced on the anode due to a charge carrier $q$ at position $r$?

$qQ_A = -q \text{ Test}_A \phi(r)$

- $\text{Test}_A \phi_A = 1$; $\text{Test}_A \phi_C = 0$
- Calculate $\text{Test}_A \phi(r)$ everywhere else
  - Could use SIMION or any electrostatics code to solve
  - Trivial for parallel-plate case
How much charge is induced on the anode due to a charge carrier \( q \) at new position \( r+dr \)?

- \( qQ_A = -q \text{Test}A \phi(r+dr) \)
- \( r+dr = r(t+dt) \) calculated by drift velocities and real electric fields
  - This would involve real fields
  - also impurity distributions, real charge carrier mobilities, etc.
How much charge is induced on the anode due to a charge carrier $q$ at position $r+dr$?

- $qQ_A = -q \text{\text{TestA}} \phi (r+dr)$

- This is equally true whether or not the charge is moving towards, away from, or for that matter, parallel to the contact. It is true for any direction.
Put in some time dependence:

At times $t$ and $t+dt$
with charge at positions $r$ and $r+dr$
what are $Q$ and $Q+dQ$?

RAMO’S THEOREM STATES:

$$\begin{align*}
qQ_A(t) &= -q^{\text{TestA}} \phi(r(t)) \\
qQ_A(t + dt) &= -q^{\text{TestA}} \phi(r(t + dt)) \\
dqQ_A(t) &= -q[^{\text{TestA}} \phi(r(t + dt)) - ^{\text{TestA}} \phi(r(t))] \\
\end{align*}$$

- (remember I set $W=1$)
How much charge change is induced on the anode due to a charge carrier $q$ at position $r+dr$ at time $t+dt$?

$$d^qQ_A(t) = -q \left[ TestA \varphi(r(t+dt)) - TestA \varphi(r(t)) \right]$$

But knowing that $E = -\nabla \varphi$,

$$[ TestA \varphi(r + dr) - TestA \varphi(r) ] = -TestA E \cdot dr$$

$$[ TestA \varphi(r(t + dt)) - TestA \varphi(r(t)) ] = -TestA E \cdot dr$$

$TestA E$ is called the **weighting field or test field**

$$d^qQ_A(t) = -q \left[ -TestA E \cdot dr \right]$$
How much current flows into the anode while the charge carrier $q$ drifts?

$$dq_{QA}(t) = q_{TestA}E(r(t)) \cdot dr$$

Divide by $dt$:

$$\frac{dq_{QA}(t)}{dt} = q_{TestA}E(r(t)) \cdot \frac{dr}{dt}$$

Our answer: The current flow INTO the anode due to the motion of $q$ is:

$$qi_A(t) = q_{TestA}E(r(t)) \cdot \nu(t)$$
How much current flows into the anode while the charge carrier q drifts?

Finally: The current flow into the anode due to the motion of q is:

\[ q_i_A(t) = \left(\frac{q}{W}\right) \text{Test}AE(r(t)) \cdot v(t) \]

Current INTO contact A is the scalar product of test field from A and velocity.
Practical points

Detailed simulations with complicated conductors and trajectories are best done by time steps $t_k$ in the test (scalar) potential solved by finite element analysis or some such thing

$$\left( q Q_A \right)_k = - q^{\text{Test}A} \varphi(r_k)$$

But for a handwaving “analysis”, the expression with the velocity and test electric field often easier to use

$$q i_A = q^{\text{Test}A} \mathbf{E} \cdot \mathbf{v}$$

(remember $W = 1$)
Practical points

And again:

\[ q_i_A = q^{TestA} E \cdot \nu \]

This is true for ANY arbitrary direction – towards the contact, away from it, parallel, whatever.
• Return to the simple 1-D problem of a parallel plate detector
• Radiation event at $x=x_0$, $t=0$
• Event liberates $e^-$ and $H^+$ with no net charge
  – $e^-$ negatively charged, $q_e = -e$
  – $H^+$ positive, $q_H = +e$
• Take constant drift velocity $v$, (believable since since $E$ is constant)
• Remember that $v_H < v_e$ sometimes by a lot
• Now we just need a test field
• Set anode to 1 and cathode to 0 (ground)
• Test potential drops at a constant rate from anode to cathode,
\[ TestA\varphi(x) = 1 - \frac{x}{\ell} \]
• The test field is
\[ TestAE = \frac{1}{\ell} \]
• Now apply that to our equation for current derived via Ramo’s theorem
• $i_A(t) = q \nu \cdot TestA E$

• The negatively charged electron induces a positive current INTO the anode

$$e i_A(t) = (-e)(-\nu_e) \frac{1}{\ell}$$

for $t < x_0/\nu_e$
\[ i_A(t) = q \mathbf{v} \cdot \text{TestA} \mathbf{E} \]

- **EVEN THOUGH THE POSTIVELY CHARGED ION IS MOVING TOWARDS THE CATHODE**, it STILL induces a positive current INTO the anode.

\[ Hi_A(t) = (+e)(v_H) \frac{1}{\ell} \]

for \( t < (\ell - x_0)/v_H \)
Negative charge-carrier

- Graphically:
  - Electron trajectory
    - Starts at $x=x_0$
    - Moves fast, speed $v_e$ in $-x$ direction, to $x=0$

\[ e_i(t) = -e v_e \cdot E \]

(for $t < \frac{x_0}{v_e}$)
Positive charge carrier

Graphically:

- **H+ trajectory**
  - Starts at \( x = x_0 \)
  - Moves slowly \( v_H \) to \( x = \ell \)

\[
H^+ \quad i_A(t) = +e \nu_e \cdot \text{TestA} E
\]
\[ i_A(t) = \sum q v \cdot \text{TestA}E \]

\[ = (-e) \left( -v_e(t) \cdot \frac{1}{\ell} \right) + (+e) \left( +v_H(t) \cdot \frac{1}{\ell} \right) \]

- Slow H's contribute low magnitude current for longer time
- Fast electrons contribute higher magnitude current for shorter time
- Black line shows sum of induced currents
- This is the current into the anode.
- This is what we wanted to know.
- This is what we use to measure the radiation.
- Total radiation-induced signal is proportional to INTEGRATED CURRENT over full charge drift time.
• Signal lasts until the last charge reaches its collecting contact

• The closer the interaction is to the anode, the longer the duration of the slow low-amplitude part
Problems with this

- Total energy loss from radiation event is proportional to the total charge.
- So you have to integrate current over the full drift time.
- Longer integration time equals more noise, potentially.

\[ \text{Drift} E = \frac{\varepsilon}{\ell} \]

\[ \text{Test} A E = \frac{1}{\ell} \]
Problems with this

- Carriers can recombine before they reach the contact
  - Random effect
  - Randomly reduces total collected charge
  - Positive carriers generally more susceptible
Problems with this

- Result: Poor energy resolution
- This wouldn’t be a very good detector.
Improved detector

- Put a conductive grid a distance $g$ in front of the anode
  - Narrow enough spacing in grid so that it very nicely approximates a flat plane
  - Charge carriers can still pass through it
Improved detector drift fields

• Put a conductive wire grid a distance \( g \) in front of the anode
  – Narrow enough wires that it very nicely approximates a flat plane

• Select voltages for the same total voltage and a constant electric field
  \( \mathcal{E}_a = \left(\frac{g}{\ell}\right) \mathcal{E} \)
  \( \mathcal{E}_b = \left(1 - \frac{g}{\ell}\right) \mathcal{E} \)
  – to keep our calculations simpler

• Drift speed, trajectories unchanged from previous case
Improved detector **test field**

- For **test field** for **anode**:
  - Ramo’s theorem says to turn off ALL the power supplies (set them to 0) except the one you are testing
  - **So set grid to ground as well**
  - $V_{\text{TestA}}, E_{\text{TestA}}$ will now be zero everywhere except between the grid and the anode
  - **Test field** $E_{\text{TestA}}$ magnitude larger: it is proportional to $1/g$ ($g$ is small) instead of $1/\ell$ ($\ell$ is bigger)
Improved detector

- Charge motion is unchanged

\[
\text{Drift } E = \frac{\varepsilon}{\ell}
\]

\[
E_a - E_b = \ell g
\]
Improved detector

- Test field is ZERO $x > g$
- Large weighting field at $x < g$

charge carriers induce Anode current only drifting through non-zero test field

- No Anode current from charge motion in yellow area

\[ \text{Test}AE = 1/g \text{ between Anode and grid} \]

\[ \text{Drift}E = \varepsilon / \ell \]

\[ \text{Test}AE = 0 \text{ in yellow area} \]
Improved detector

Anode sees no induced current from charges where test field is zero

TestA\( E = \frac{1}{g} \) between Anode and grid

\[ i_A(t) \]

\[ E = \frac{\varepsilon}{\ell} \]

Drift
Improved detector

- H+ induces no anode current
- e- induces large anode current only while x<g

TestA $E = 1/g$ between Anode and grid

Drift $E = \varepsilon/\ell$

$\ell$

TestA $E = 0$

$\ell$

$E_a$, $E_b$
Improved detector

This gives you a:

- Faster signal
- Higher amplitude
- Not affected by recombination

- Compared to bare-anode signal, Signal from gridded detector is just plain better

\[
\text{Drift} E = \varepsilon / \ell \]

This gives you:

- Faster signal
- Higher amplitude
- Not affected by recombination
Improved detector

- The grid is called a Frisch Grid
- It improves detector performance by increasing signal amplitude, reducing integration noise, and shielding recombination
- We showed how it does this with Ramo’s Theorem

- Turn the problem around: We can use Ramo’s theorem, weighting field concept to make detectors better
Handwaving:
What about the coaxial detector?

- Consider core contact, n-type detector
  - positive voltage on Li inner core
  - $R_o - r_i \sim 2.5 \text{ cm}$
  - $v_e \sim 2.5 \times 10^7 \text{ cm/s}$, $v_h \sim 1.7 \times 10^7 \text{ cm/s}$
  - Drift velocities saturate at a nearly constant maximum at typically HPGe operating conditions
  - For our exercise assume drift velocity is constant
Handwaving:
What about the coaxial detector?

- Ramo weighting $E_w$ field is what we expect for a coaxial capacitor: except for right at the bottom,
  \[ \text{Test } E_w \propto \frac{1}{r} \]

- Charge-collecting preamps

- Output voltage is integral of current into preamp i.e. out of detector

- So output signal
  \[ V \propto - \int \vec{E} \cdot d\vec{r} \]

- Integration by handwaving
Handwaving: What about the coaxial detector?

- Case A: Interaction near core
  - Electrons reach core immediately – almost no induced charge (!)
  - Holes drift for 170 ns
  - Initial current on core (induced current): since $r$ is small and $E_w \sim 1/r$ is large, current from holes is large
    - So initially, slope $dq/dt$ is large
  - As holes drift to outside, $E_w \sim 1/r$ decreases
    - So slope $dq/dt$ decreases
  - Results in the convex signal from earlier slides.
Handwaving: What about the coaxial detector?

• Case B: interaction near outside
  – Holes reach outside almost immediately – very little induced charge
  – Electrons drift for 100 ns
  – Initial current on core (induced current) from holes is small
    • Since $r$ is large and $E_w \sim 1/r$ small
      • So slope $dq/dt$ is small
  – Current increases as holes reach outside ($r$ increases)
  – Results in the concave signal from earlier slides.
Handwaving looks a lot like real measured signals

- “Handwaving” was for current INTO contacts
- Oscilloscope traces are proportional to current OUT OF contacts
- The hardest part of this is keeping track of signs – especially current direction
Summary
or what I hope you got out of this

• Signals from radiation-induced events on electronic detectors evolve with time while charge carriers are moving
• Time dependence of signal depends on charge carrier trajectory and configuration of conductors
• Part of calculating the signal shape requires a theoretical *weighting field* or test field calculated with the electrode of interest at a potential of 1 and the remaining conductors set to ground
• Once the principle is established, simple arguments using weighting fields are often adequate to explain detector signals and to design better detectors.
Further thoughts

Currents to Conductors Induced by a Moving Point Charge

W. Shockley
Bell Telephone Laboratories, Inc., New York, N. Y.
(Received May 14, 1938)

• Proportional gas counters and some semiconductors like CdZnTe have very slow positive carriers prone to recombination – Frisch grids are essential for those
• Small and point-like contacts have very intense weighting fields near them – almost as good as having a Frisch grid
• Early applications of these concepts were to vacuum tubes: Phys. Rev. 9, 645 (1938!)