# Nuclear Shell Model & Alpha Decay

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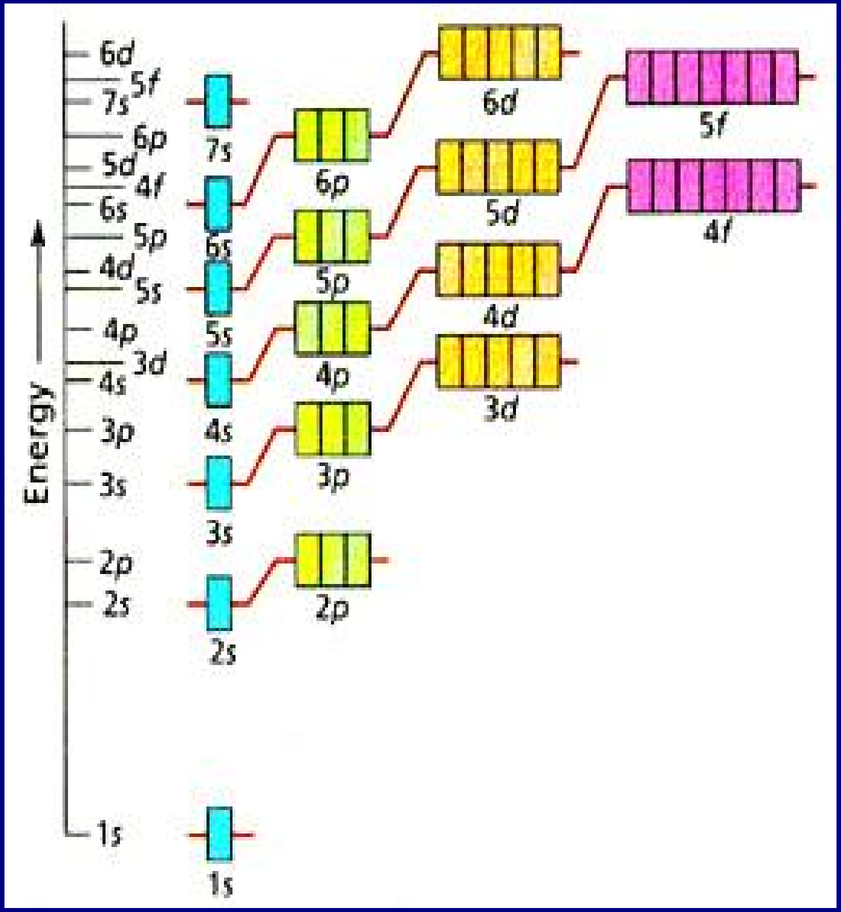
TRIUMF

## Nuclear Shell Model

Electrons in atoms occupy well-defined shells of discrete, well-separated energy.  Do nucleons inside a nucleus do the same, or not?




Evidence for electron shells in atoms:   sudden jumps in atomic properties as a shell gets filled up, e.g.  atomic radius, ionization energy, chemical properties.

atomic radius

atomic ionization energy

from Krane, Introductory Nuclear Physics



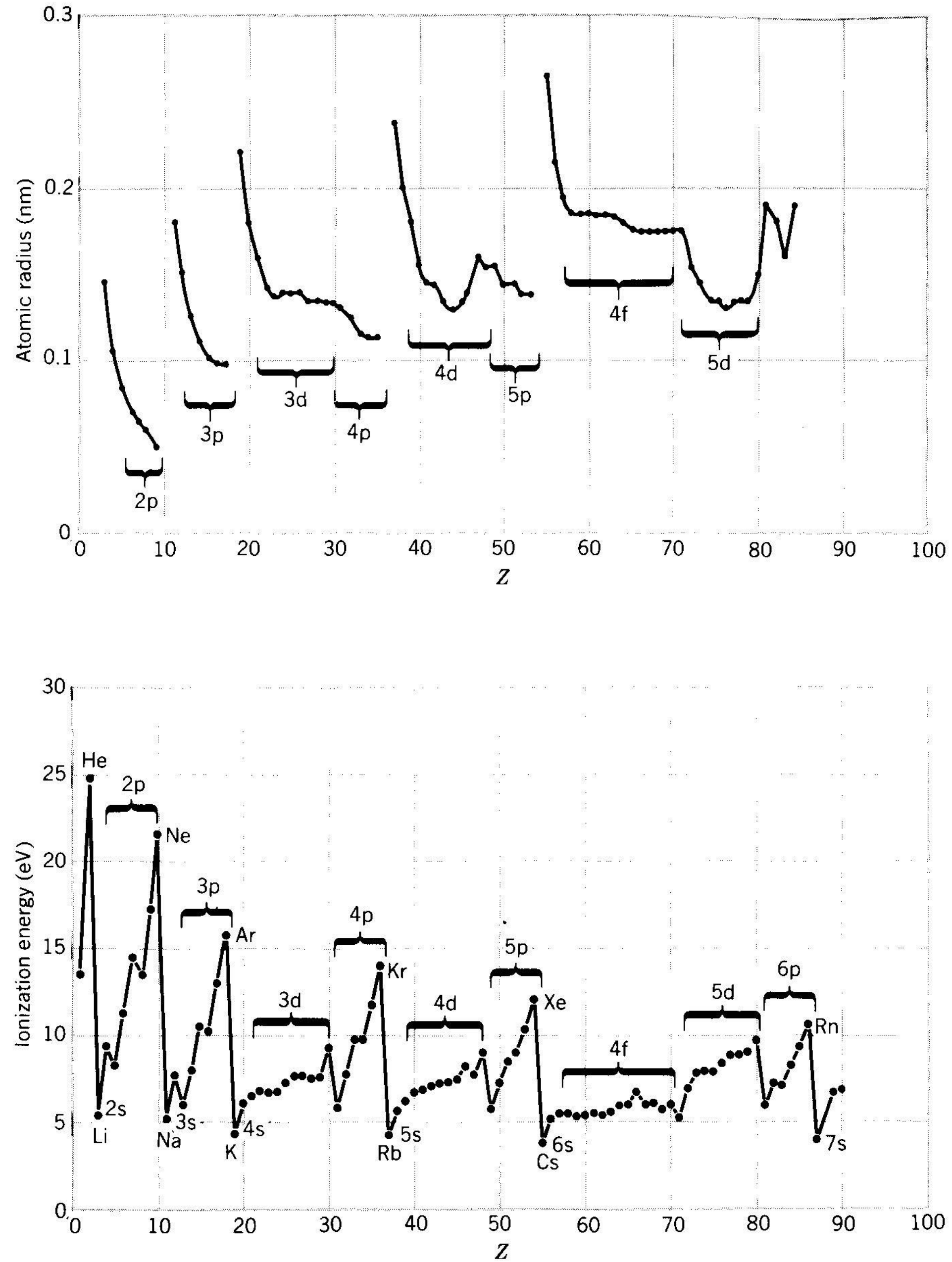
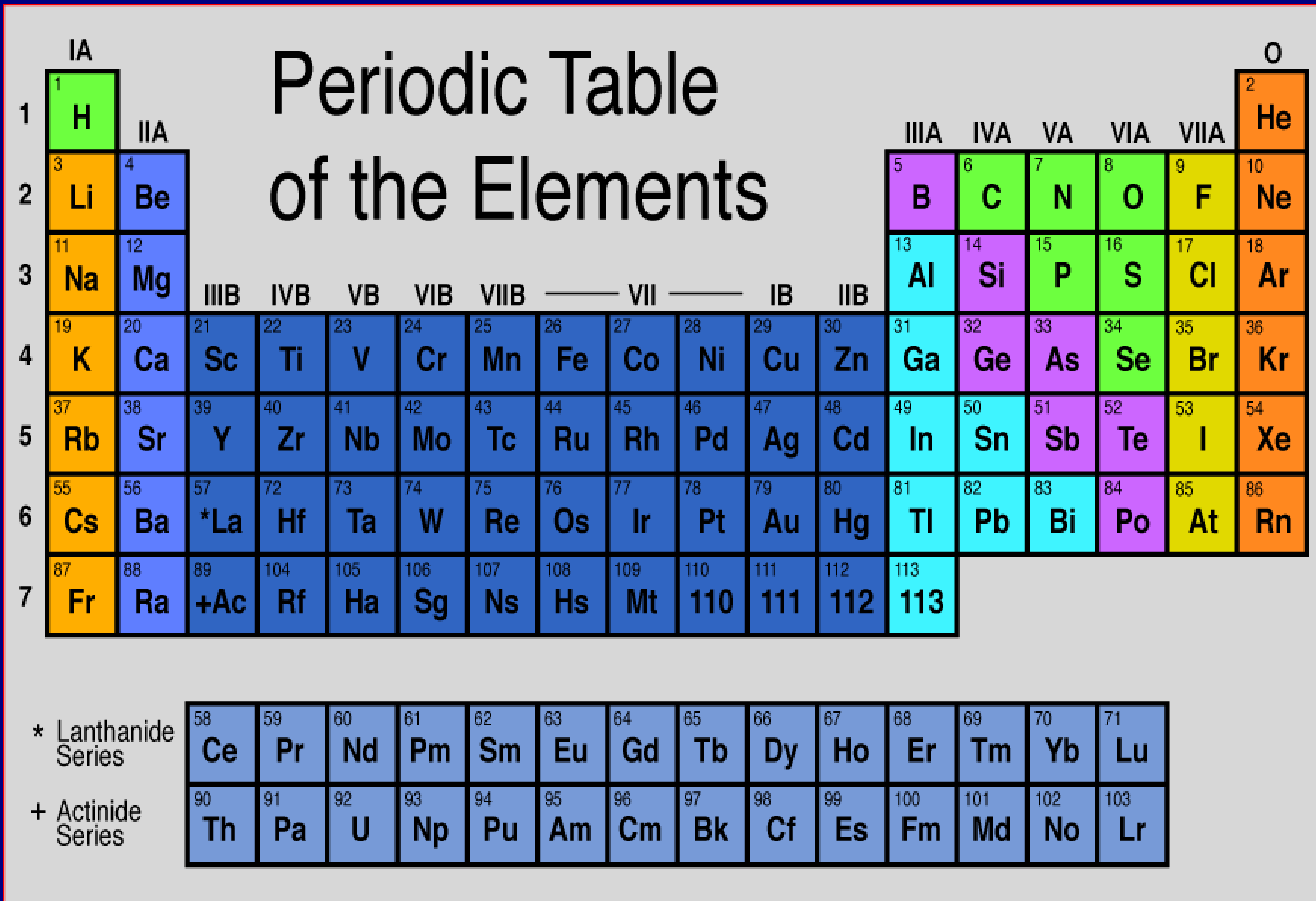



**Figure 5.1** Atomic radius (top) and ionization energy (bottom) of the elements. The smooth variations in these properties correspond to the gradual filling of an atomic shell, and the sudden jumps show transitions to the next shell.

chemical reactivity



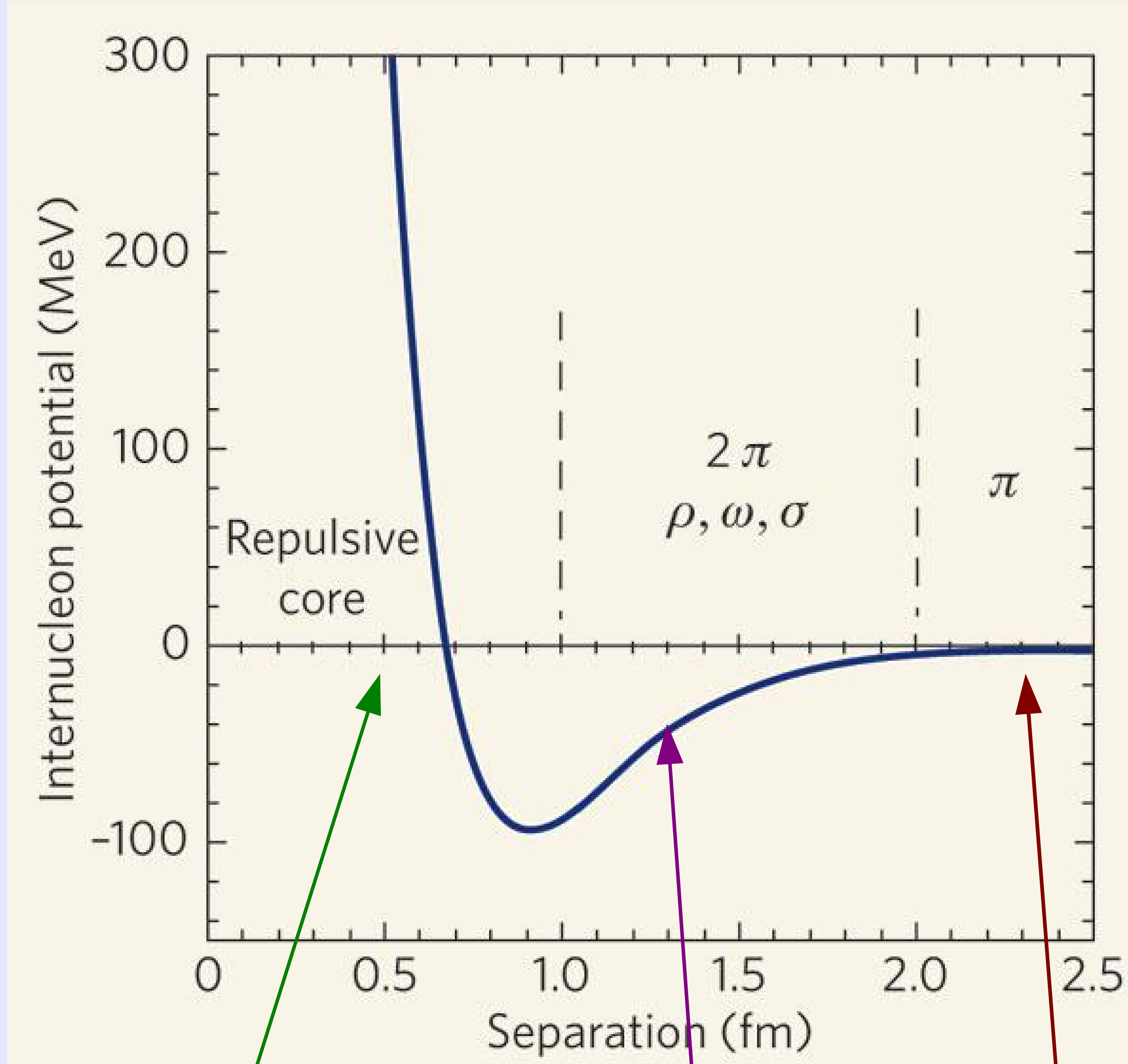
Not evident a priori that nuclei <u>should</u> show shell structure. Why not?

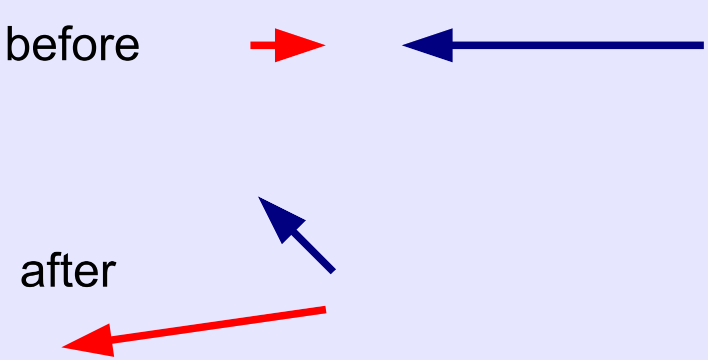
1. Success of liquid drop model in predicting nuclear binding energies. Liquid drops have smooth behaviour with increasing size, and do not exhibit jumps.

2. No obvious center for nucleons to orbit around, unlike electrons in an atom.

**<u>3rd objection to a shell model</u>**

Strong repulsive core in the nucleon-nucleon potential should scatter the nucleons far out of their orbits -- nucleons should not have a well-defined energy, but should behave more like molecules in a gas, colliding and exchanging energy with each other.




overlap of 3-quark bags; complicated short-range behaviour

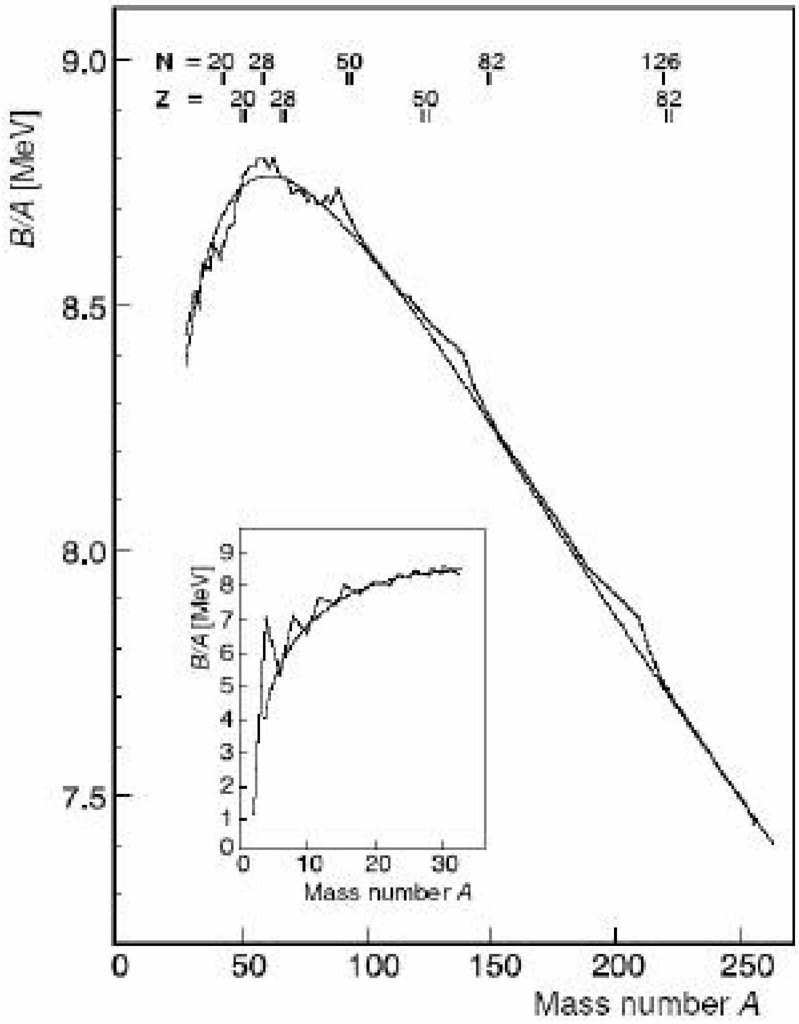
two pion and heavy meson exchange

one pion exchange

Many theoretical reasons were given why nuclei should not show any shell structure ....

but experiment says otherwise!

# BINDING ENERGIES & EXPERIMENT




- Binding energy per nucleon of nuclei with even mass number A.
- Solid line corresponds to semi-empirical mass formula.
- Relatively large deviations for small A.
- For large A somewhat stronger binding at certain Z and N. These so-called 'magic numbers' will be discussed when we consider the shell model.

2-neutron separation energy  in nuclei
-- analogous
to ionization energy
in atoms

Note jumps at
nucleon numbers
8, 20, 28, 50, 82, 126

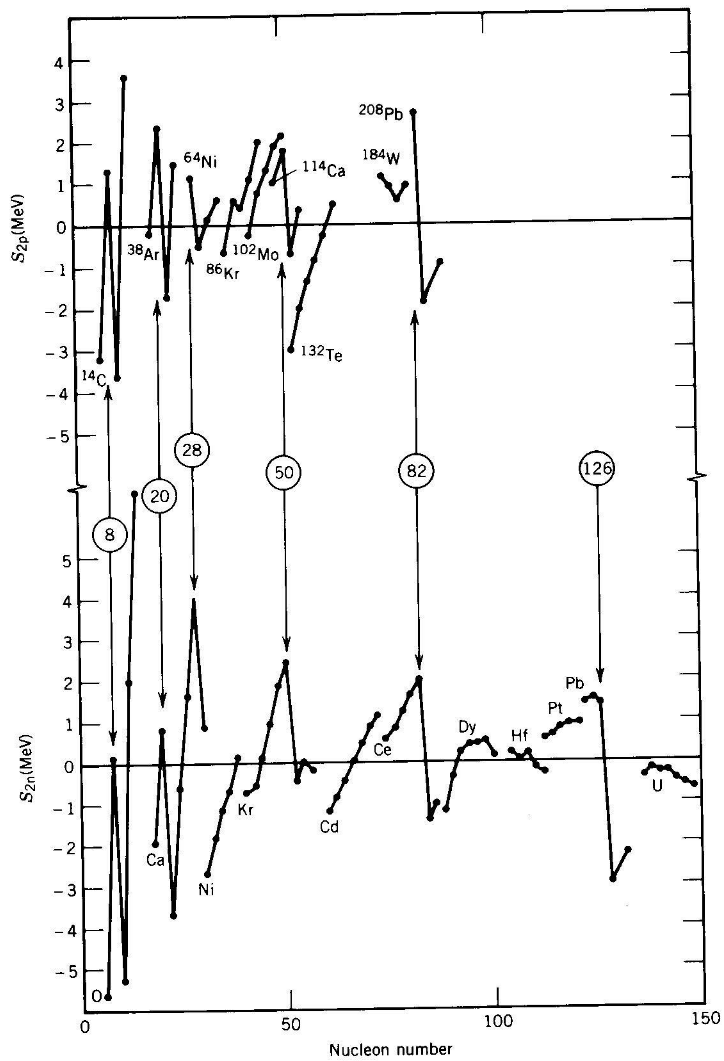



**Figure 5.2** (Top) Two-proton separation energies of sequences of isotones (constant $N$). The lowest $Z$ member of each sequence is noted. (Bottom) Two-neutron separation energies of sequences of isotopes. The sudden changes at the indicated "magic numbers" are apparent. The data plotted are differences between the measured values and the predictions of the semiempirical mass formula. Measured values are from the 1977 atomic mass tables (A. H. Wapstra and K. Bos, *Atomic Data and Nuclear Data Tables* **19**, 215 (1977)).

from Krane,
Introductory
Nuclear Physics

abundances
of even-even
nuclides

local maxima
at 50, 82, 126

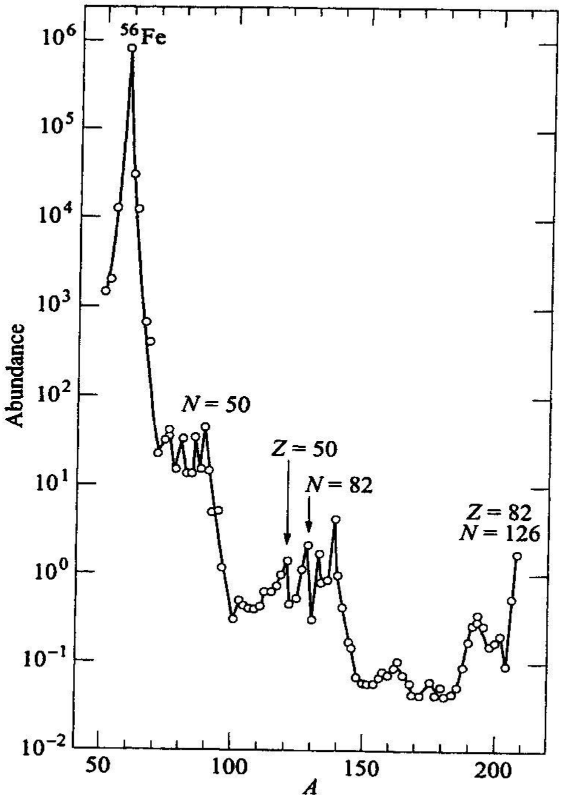



**Fig. 17.4** Relative abundances of different even–even nuclides, plotted as a function of $A$. The abundances are measured relative to Si, with $H(\mathrm{Si}) = 10^6$ (Cameron 1968).

neutron capture
cross section
--
analogous to
chemical
reactivity

dips at N=20, 50,
82, 126

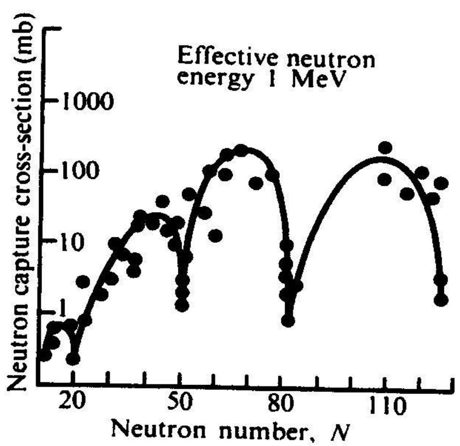



**Fig. 17.5** Neutron capture cross-sections corrected to an effective neutron energy of 1 MeV showing minima at $N = 20$, 50, 82 and 126. This shows that structures with these numbers of neutrons are very stable and a further neutron is weakly bound (Burge 1977).

from
Gadioli &
Gadioli,
Intro.
Nuclear
Physics

alpha particle emission energy vs neutron number of parent nucleus

max for N=128 parent (N=126 daughter)

neutron capture cross section (repeat)

nuclear charge radius

minima at N=20, 28, 50, 82, 126

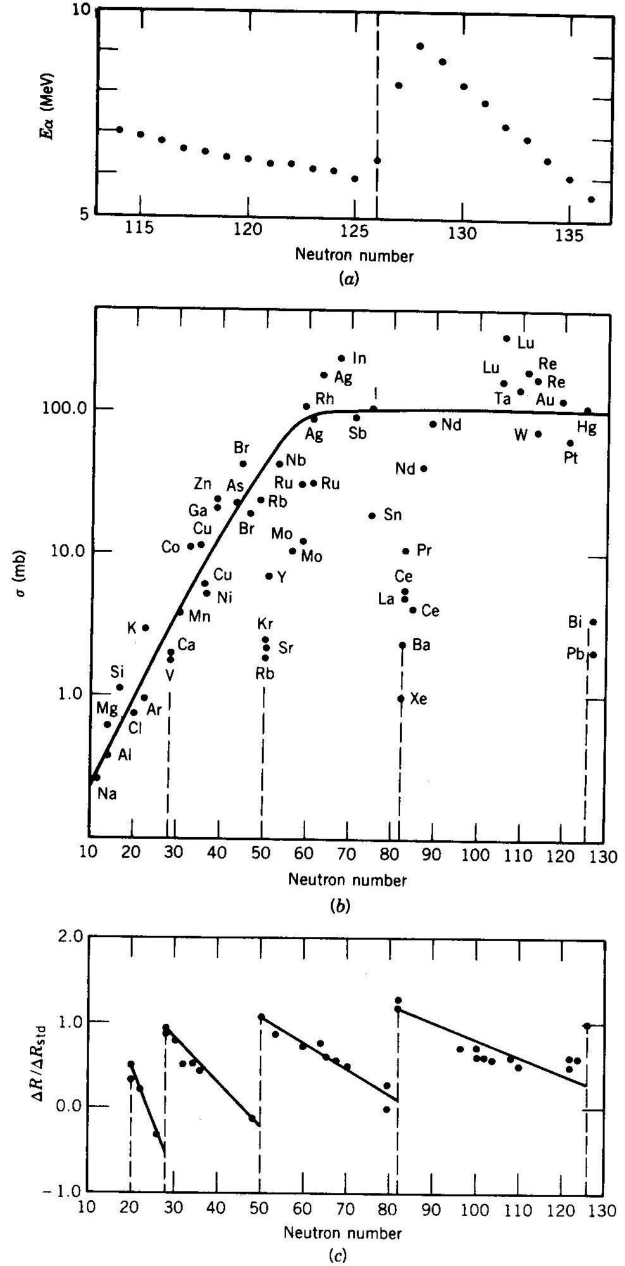



**Figure 5.3** Additional evidence for nuclear shell structure. (*a*) Energies of $\alpha$ particles emitted by isotopes of Rn. Note the sudden increase when the *daughter* has $N = 126$ (i.e., when the parent has $N = 128$). If the daughter nucleus is more tightly bound, the $\alpha$ decay is able to carry away more energy. (*b*) Neutron-capture cross sections of various nuclei. Note the decreases by roughly two orders of magnitude near $N = 50$, 82, and 126. (*c*) Change in the nuclear charge radius when $\Delta N = 2$. Note the sudden jumps at 20, 28, 50, 82, and 126 and compare with Figure 5.1. To emphasize the shell effects, the radius difference $\Delta R$ has been divided by the standard $\Delta R$ expected from the $A^{1/3}$ dependence. From E. B. Shera et al., *Phys. Rev. C* **14**, 731 (1976).

from Krane, Introductory Nuclear Physics

Experimental data indicates local maxima in binding energy, local minima in charge radius, minima in neutron absorption when proton or neutron numbers are one of the following “magic numbers”

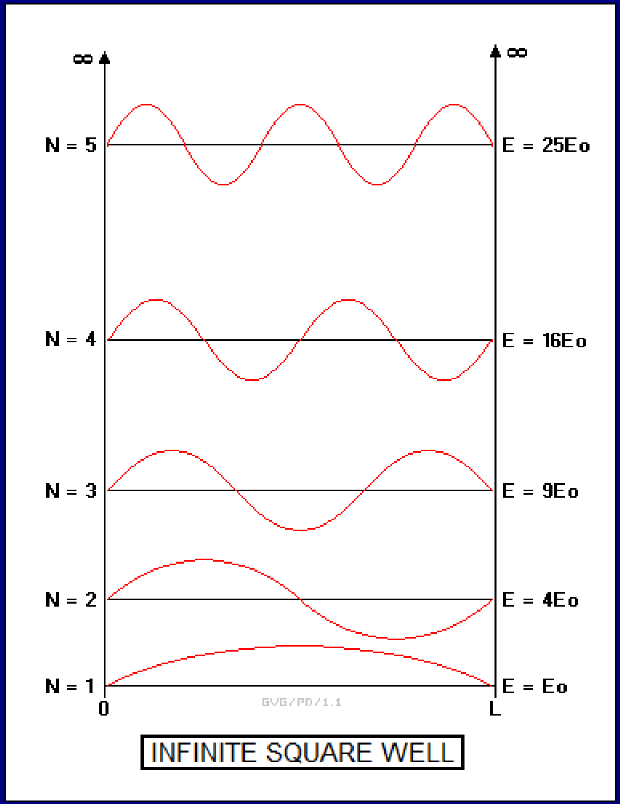
2, 8, 20, 28, 50, 82, 126

These numbers mark the shell closures for nucleons, analogous to electron shell closures for atoms.

We're physicists, we don't believe in magic!  So where do these “magic numbers” come from?

It must be related to the shape of the potential well that the nucleons sit in...

Reminder: 1D infinite square well $V(x) = 0$ for $0 < x < L$, $V(x) = \infty$ otherwise




wavefunction is like a guitar string constrained at both ends (wavefunction excluded from region $x<0$ and $x>L$ by infinitely high potential barrier).

wavelength $\lambda = 2L/n$  $n=1, 2, 3, \ldots$

De Broglie: momentum $p = h/\lambda = nh/2L$

Potential energy $V(x)=0$ inside well

$E = p^2/2M = n^2h^2 / (8ML^2)$

Greg Goebel, www.vectorsite.net/tpqm_02.html

A different shape for the potential well would result in a different set of level spacings, e.g.

infinite square well potential

harmonic oscillator potential

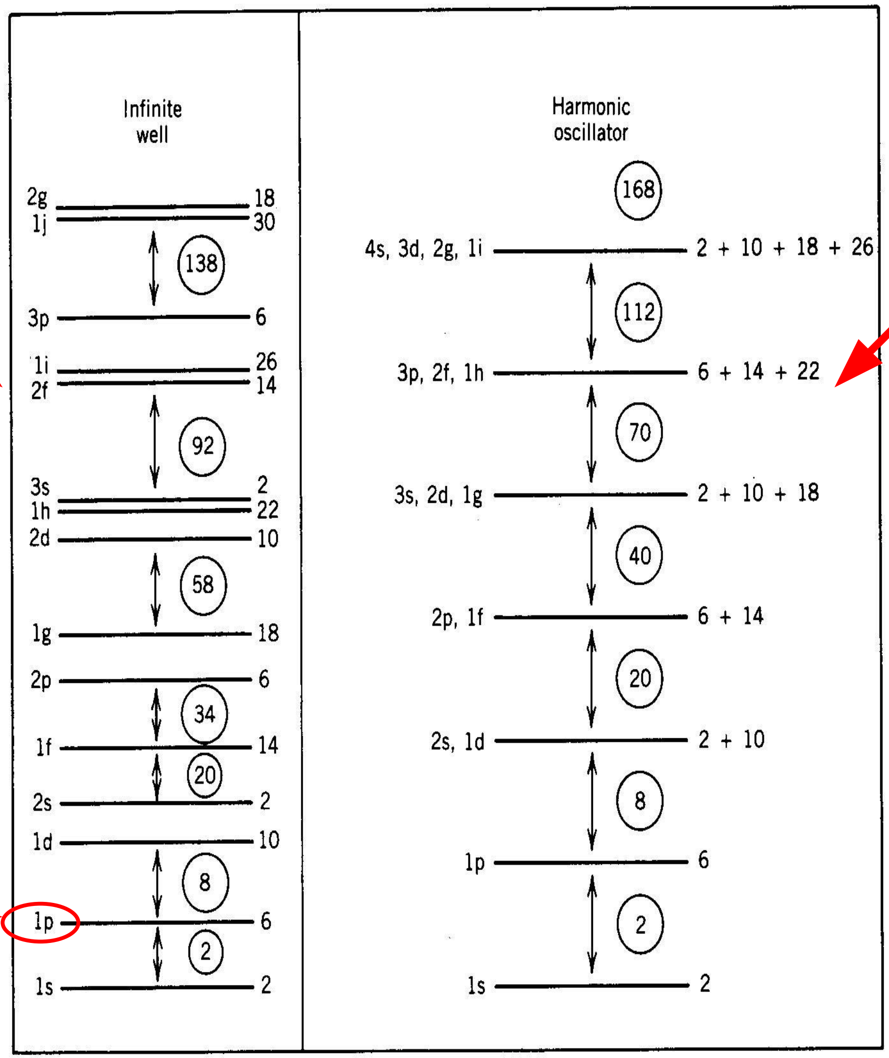



**Figure 5.4** Shell structure obtained with infinite well and harmonic oscillator potentials. The capacity of each level is indicated to its right. Large gaps occur between the levels, which we associate with closed shells. The circled numbers indicate the total number of nucleons at each shell closure.

Notation
s means L=0
p means L=1
d L=2
f L=3

p state can have 2L+1 =3 substates

each substate can have spin up or spin down

Thus 6 protons and 6 neutrons in 1p level.

Krane, Introductory Nuclear Physics

From lecture 1, we learned that the charge distribution of a nucleus looks like the figure below, and since the nuclear forces are short range, the nuclear potential must also be shaped like this:

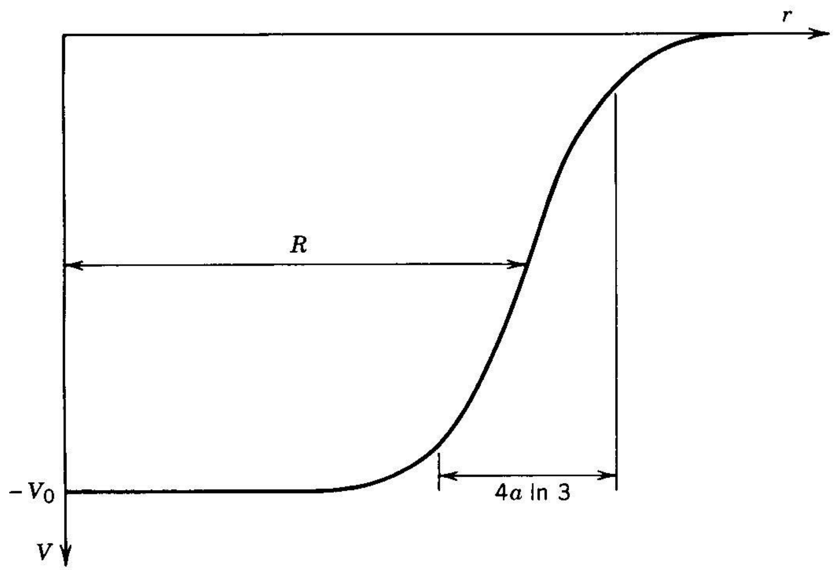
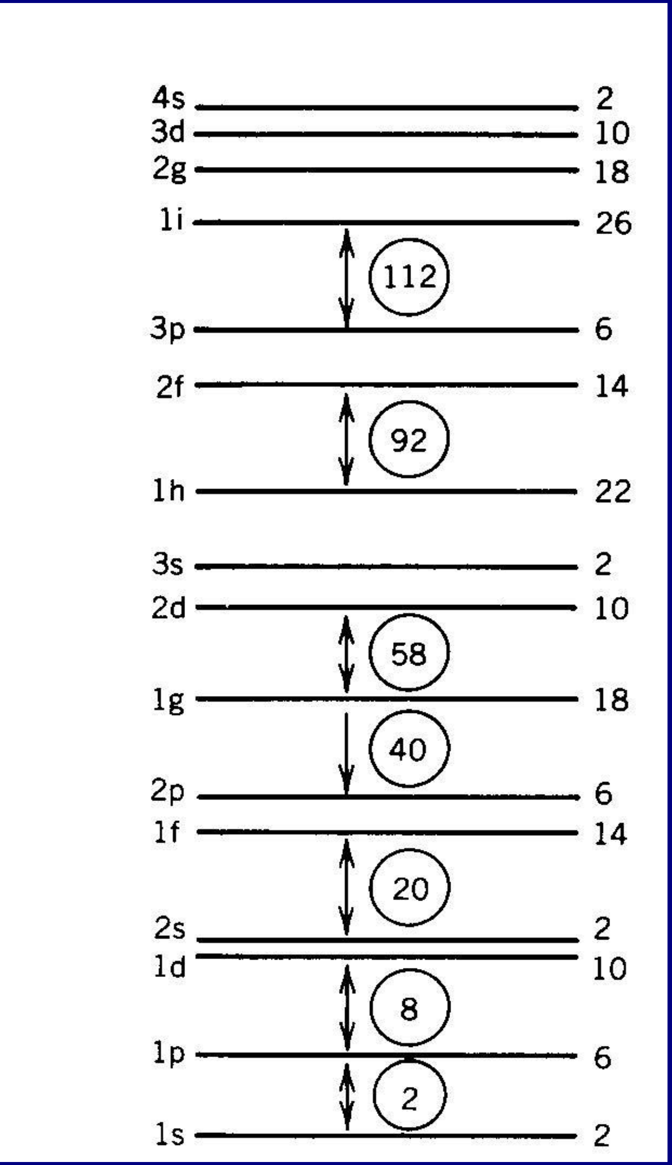



**Figure 5.5** A realistic form for the shell-model potential. The "skin thickness" $4a \ln 3$ is the distance over which the potential changes from $0.9V_0$ to $0.1V_0$.

from Krane, Introductory Nuclear Physics

But this shape of potential doesn't give the right energy level spacings either




Empirical magic numbers
2, 8, 20, 28, 50, 82, 126

No reasonable shape of nuclear potential seems to work ...

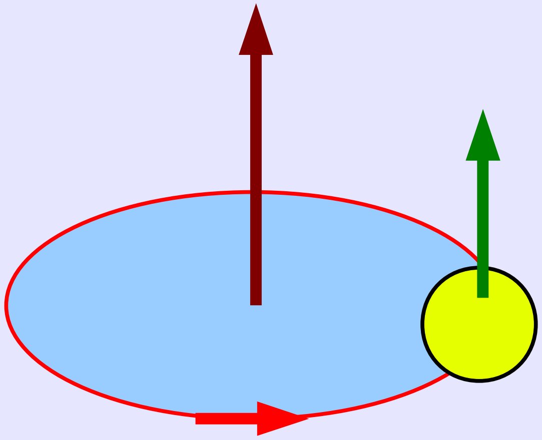
Fermi's suggestion: “Any evidence for a spin-orbit force?”

$V(r) = V_0(r) + V_{LS}\, \vec{\mathbf{L}} \cdot \vec{\mathbf{S}}$

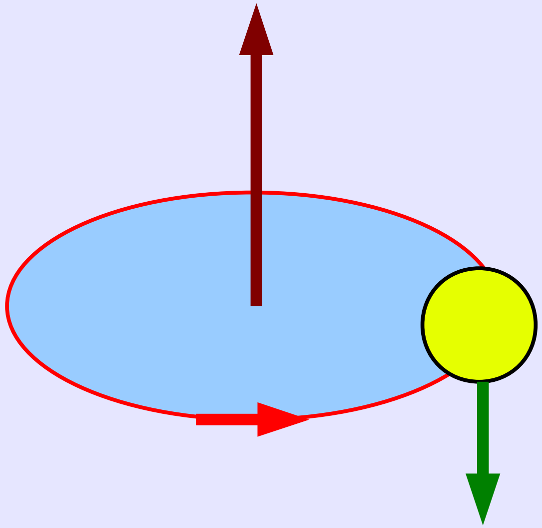
$\vec{\mathbf{L}}$ = orbital angular momentum

$\vec{\mathbf{S}}$ = spin angular momentum

where the potentials $V_0$ and $V_{LS}$ are negative (i.e. attractive)

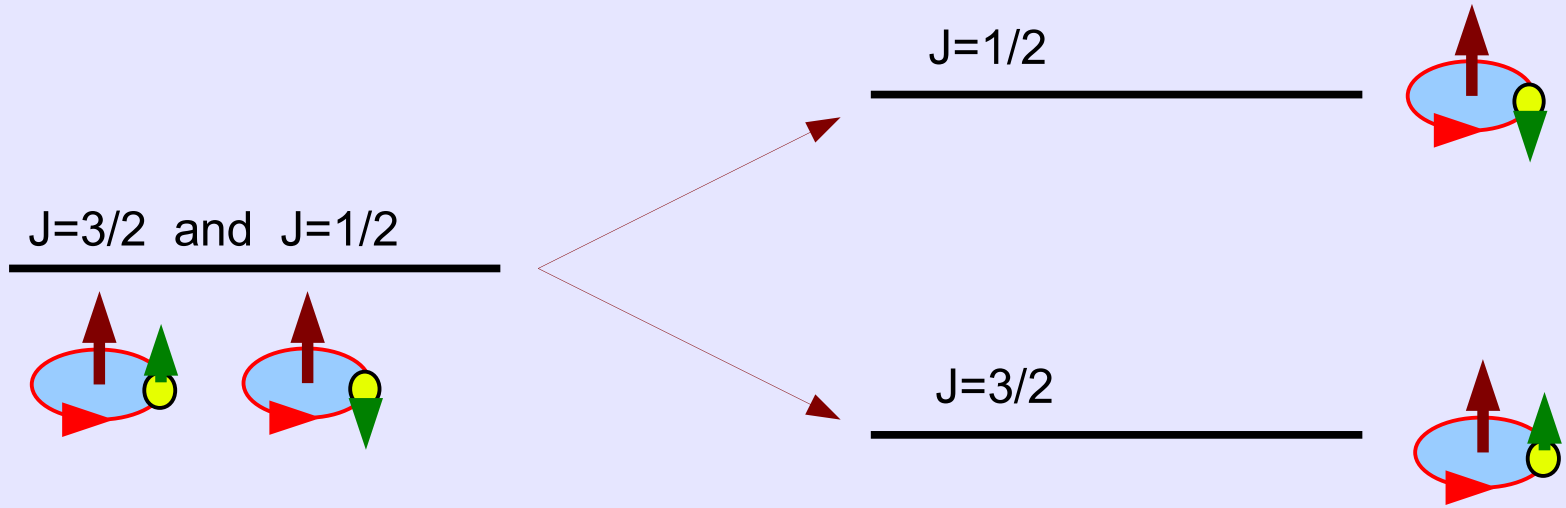


L and S parallel
deeper attractive potential



L and S anti-parallel
shallower attractive potential

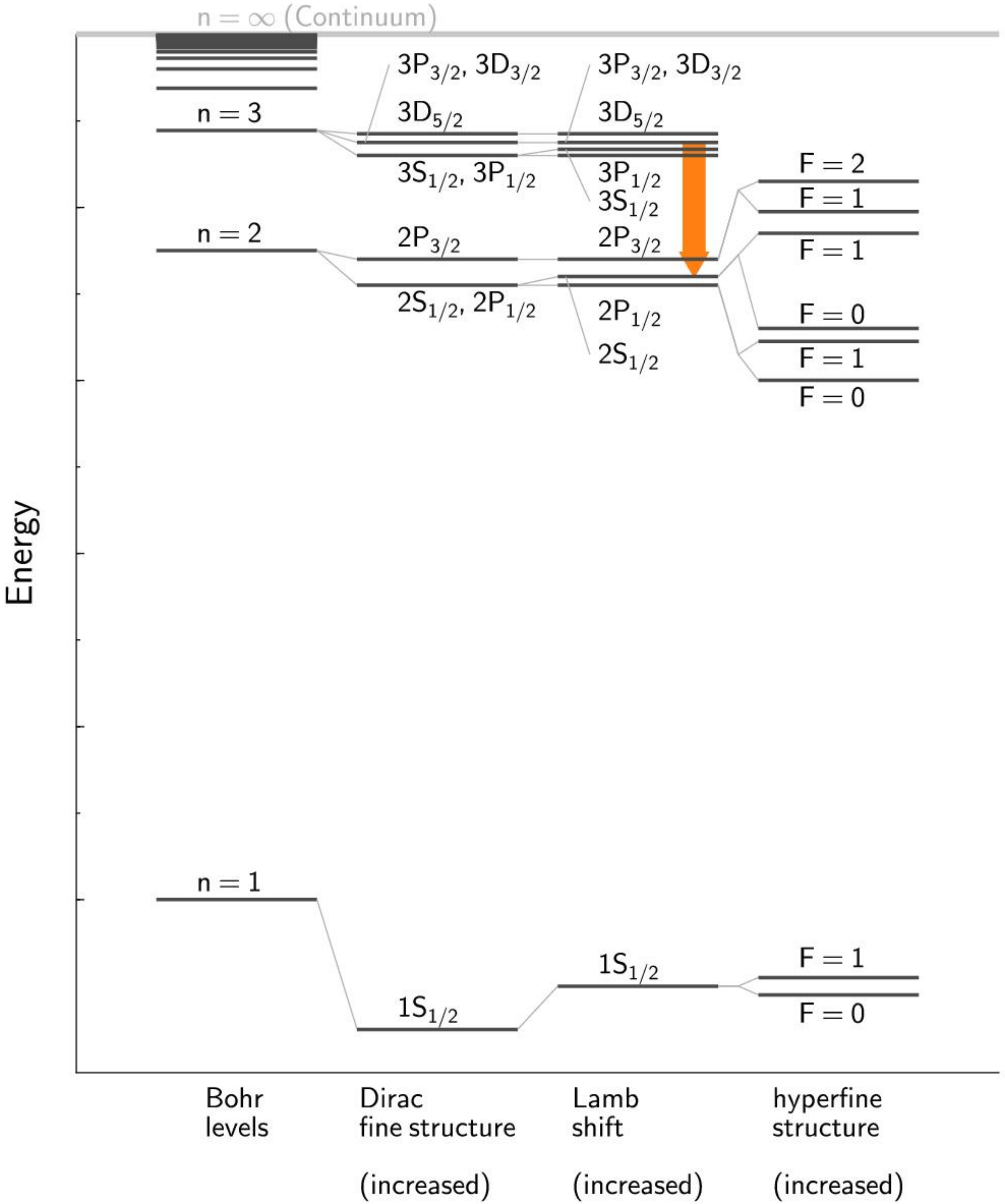
So, for example, if L=1 S=½ then by the rule of angular momentum addition in quantum mechanics, the total angular momentum J = 1 + ½ = 3/2 (parallel) or J = 1 - ½ = ½ (anti-parallel). In the presence of a spin-orbit force, these two levels are split; the larger the L value, the larger the splitting.




In the H atom, this is a small effect which results in the "fine structure" of the hydrogen atom spectral lines. In nuclear physics, the spin-orbit effect is large enough to shift the energy levels up into the next major shell, thereby changing the location of the large gaps.

Hydrogen atom:

Small spin-orbit splitting
(greatly exaggerated; actual splitting of $2P_{3/2}$ and $2P_{1/2}$ is only 0.000045 eV compared to the ionization energy of 3.4 eV, i.e. ~ 1 part in $10^5$)



With a nuclear spin-orbit force, the levels now match what is observed experimentally:

1g level
is split so
much that
the two
components
now lie
in different
shells

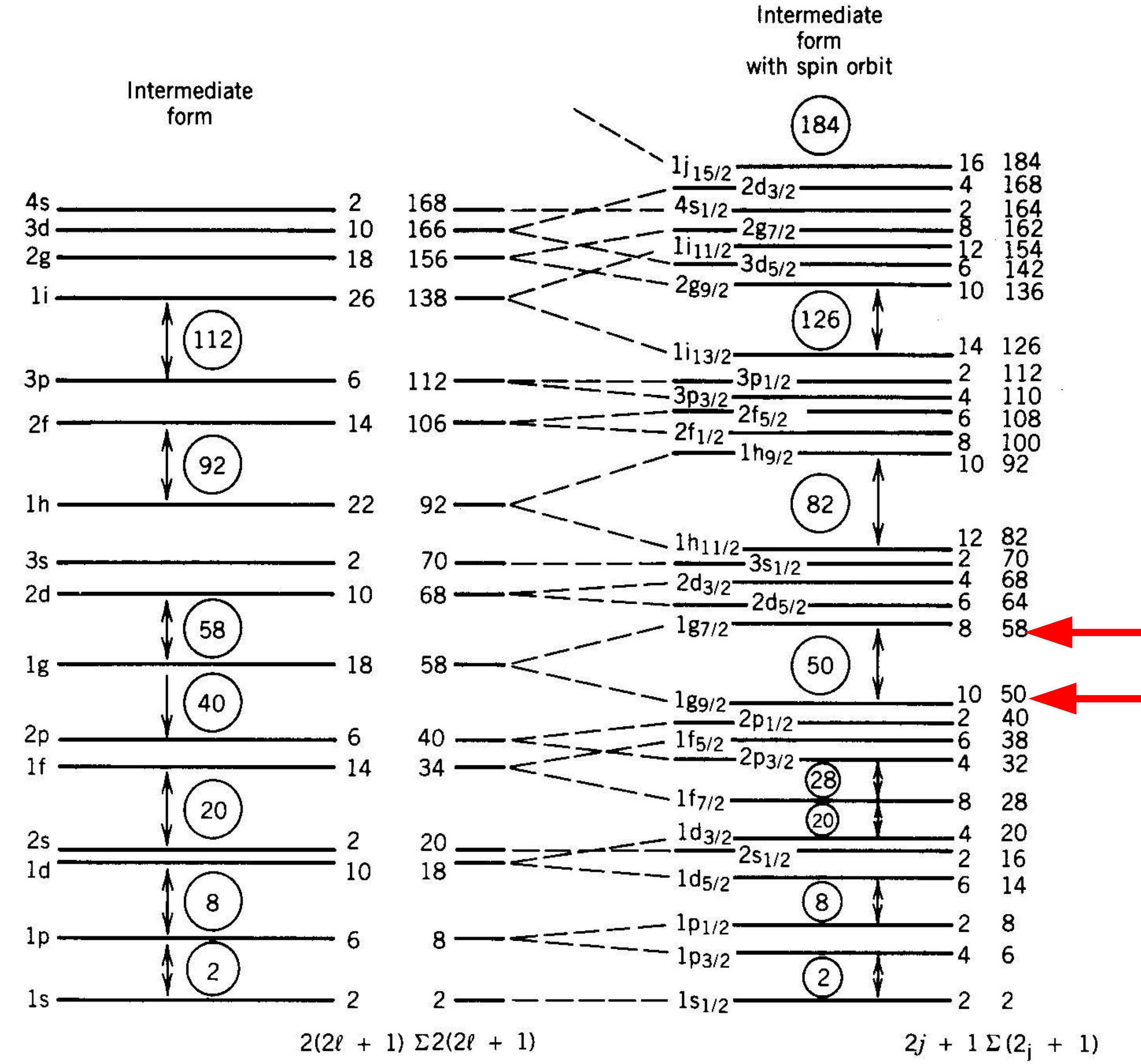
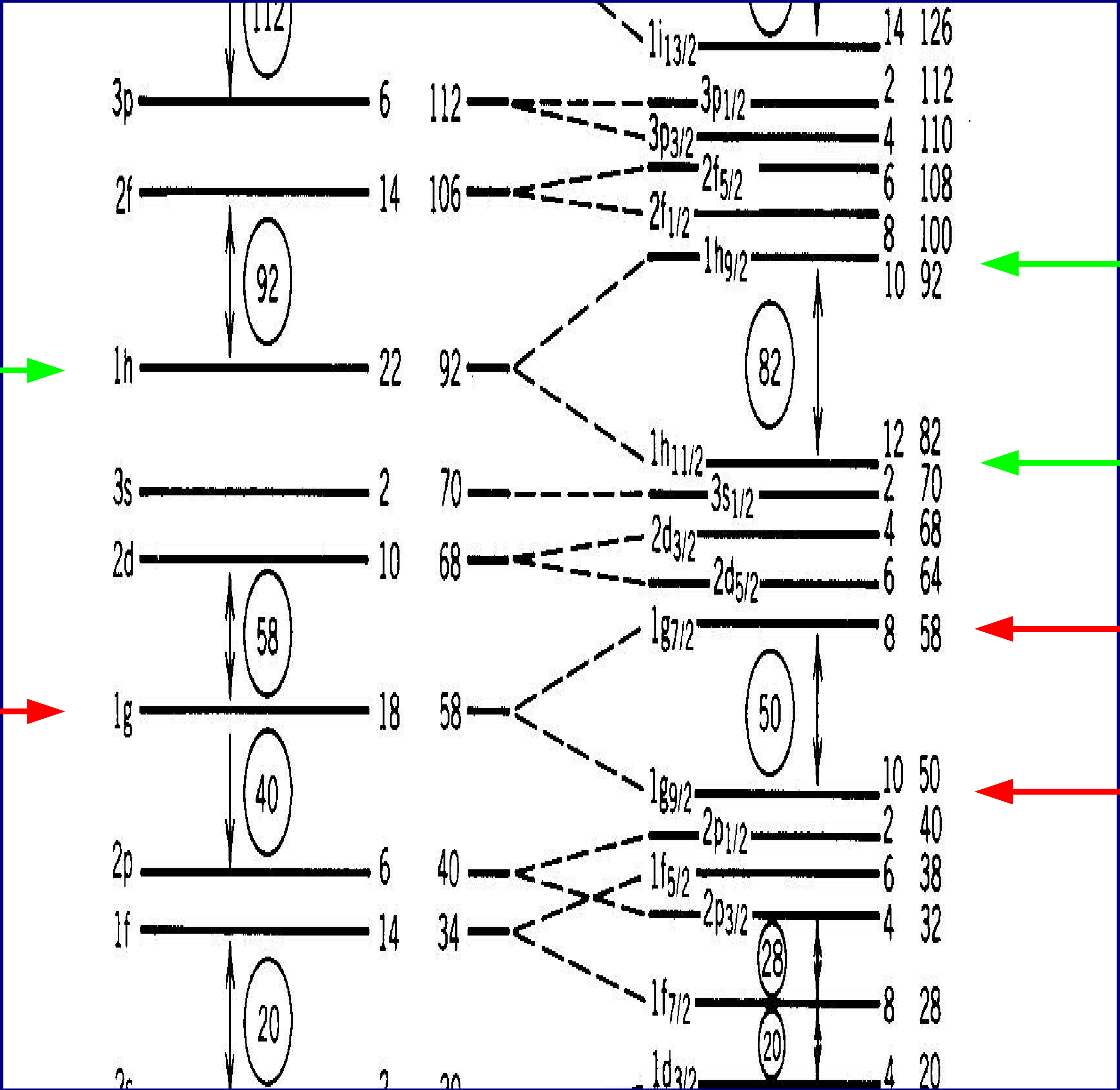



**Figure 5.6** At the left are the energy levels calculated with the potential of Figure 5.5. To the right of each level are shown its capacity and the cumulative number of nucleons up to that level. The right side of the figure shows the effect of the spin-orbit interaction, which splits the levels with $\ell > 0$ into two new levels. The shell effect is quite apparent, and the magic numbers are exactly reproduced.



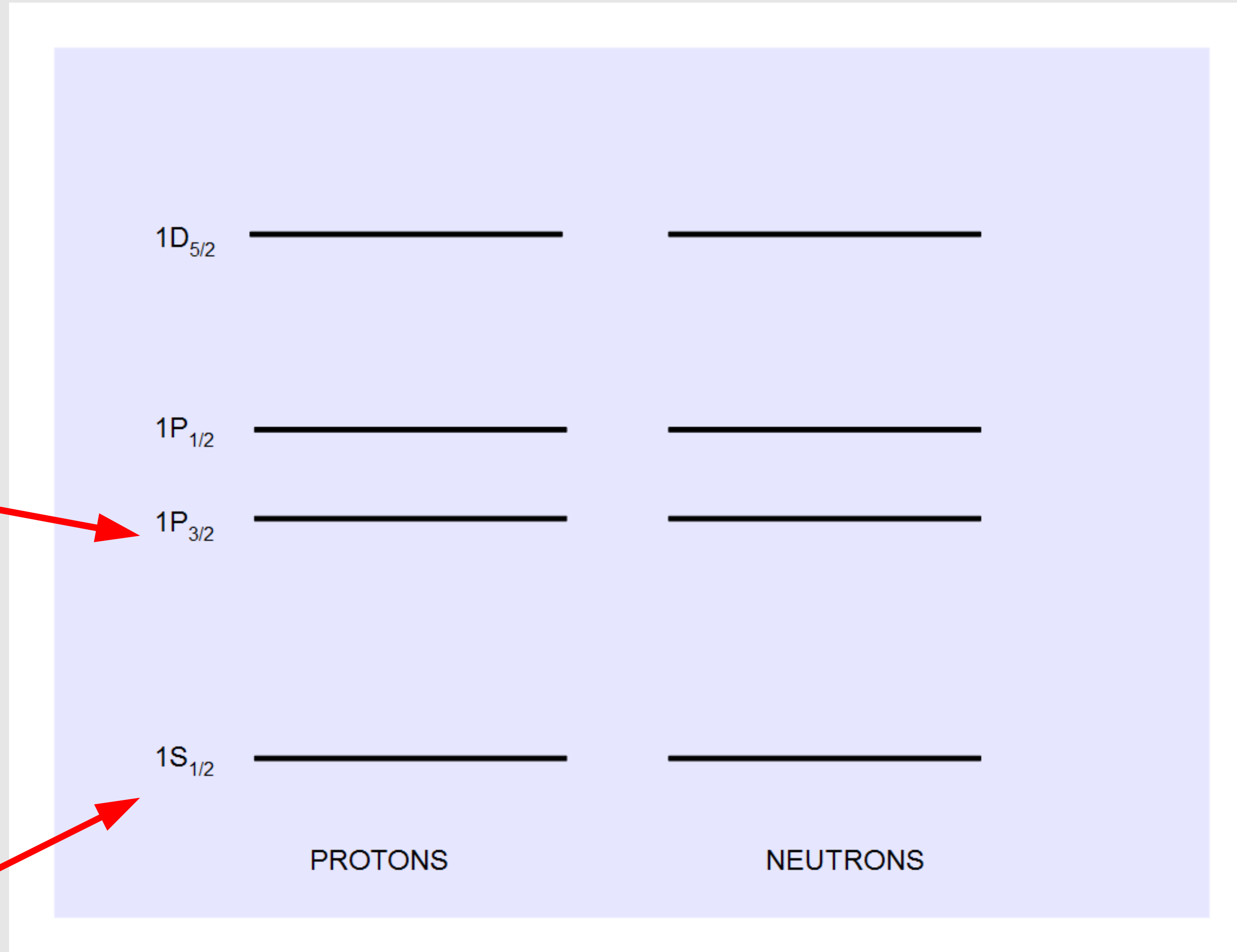
3p 6 112
2f 14 106
1h 22 92
3s 2 70
2d 10 68
1g 18 58
2p 6 40
1f 14 34
112
92
58
40
20
$1i_{13/2}$ 14 126
$3p_{1/2}$ 2 112
$3p_{3/2}$ 4 110
$2f_{5/2}$ 6 108
$2f_{7/2}$ 8 100
$1h_{9/2}$ 10 92
82
$1h_{11/2}$ 12 82
$3s_{1/2}$ 2 70
$2d_{3/2}$ 4 68
$2d_{5/2}$ 6 64
$1g_{7/2}$ 8 58
50
$1g_{9/2}$ 10 50
$2p_{1/2}$ 2 40
$1f_{5/2}$ 6 38
$2p_{3/2}$ 4 32
28
$1f_{7/2}$ 8 28
20
$1d_{3/2}$ 4 20

Maria Goeppert-Mayer
Hans Jensen
Nobel Prize in Physics, 1963

Now we can build up the shell structure of nuclei, just like we fill up electron shells in chemistry class, only now there are separate proton and neutron shells.

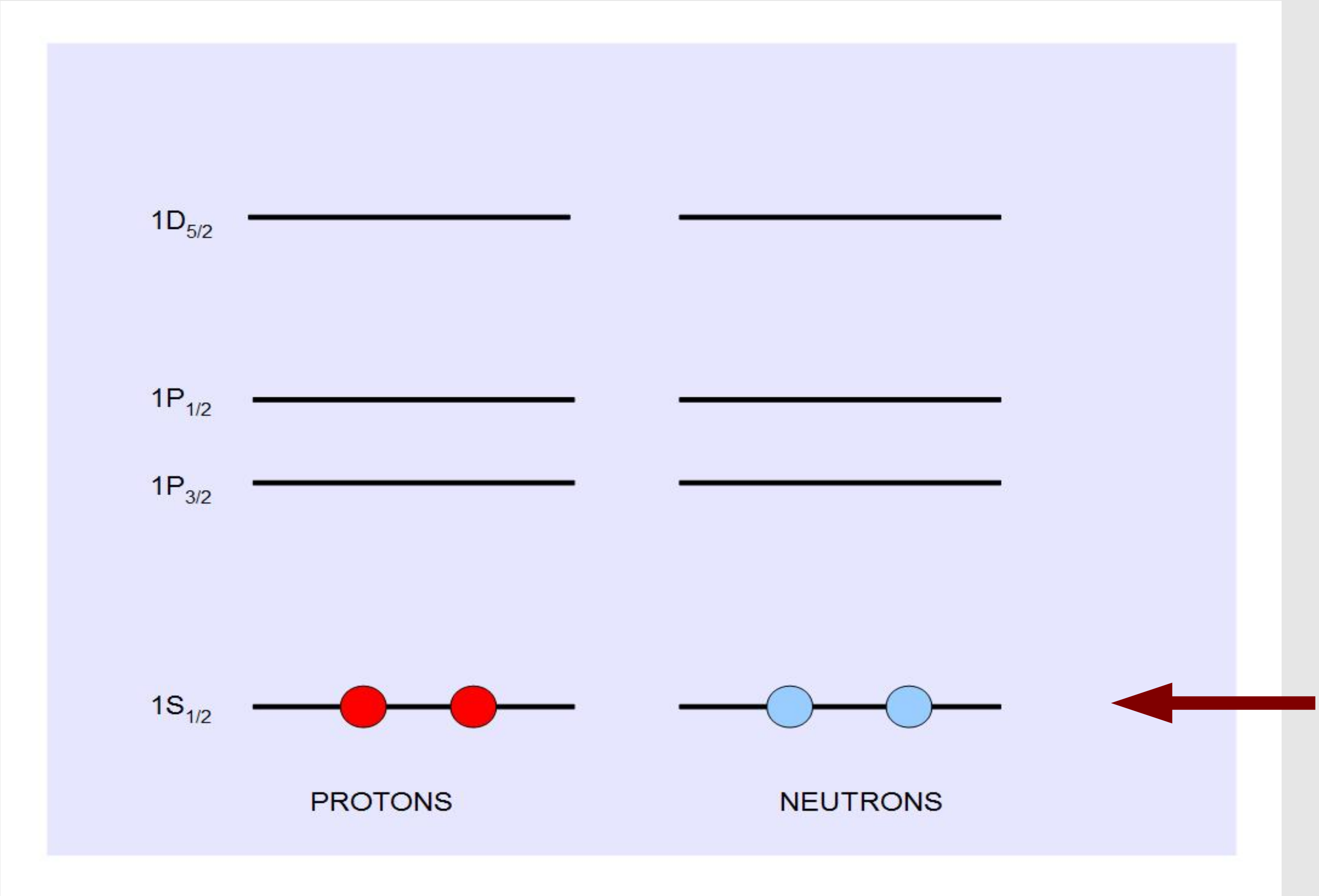
J=3/2 so there are
4 possible orientations
namely
$J_z$=3/2, 1/2, -1/2, -3/2
Therefore maximum
of 4 nucleons
in this sub-shell




J=1/2 so there are
2 possible orientations,
namely $J_z$ = +1/2, -1/2
Therefore maximum
of 2 nucleons
in this sub- shell.

In general, shell with angular momentum J can hold up to 2J+1 nucleons of each type.

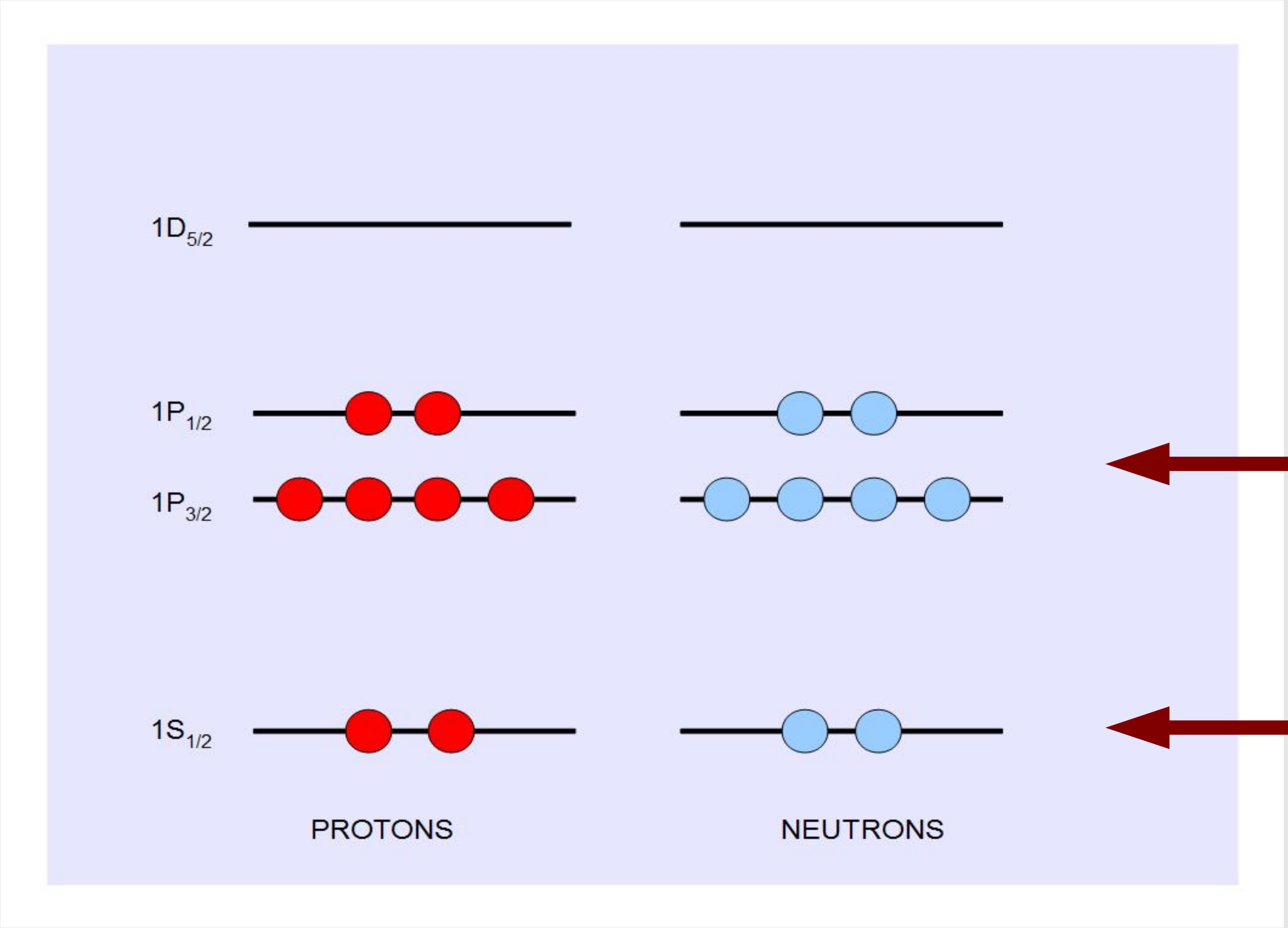
$^4$He




filled 1S shell corresponding to magic no. 2

so He-4 is a closed-shell nucleus with extra high binding energy, extra-small radius, extra-low reaction probability

# $^{16}O$



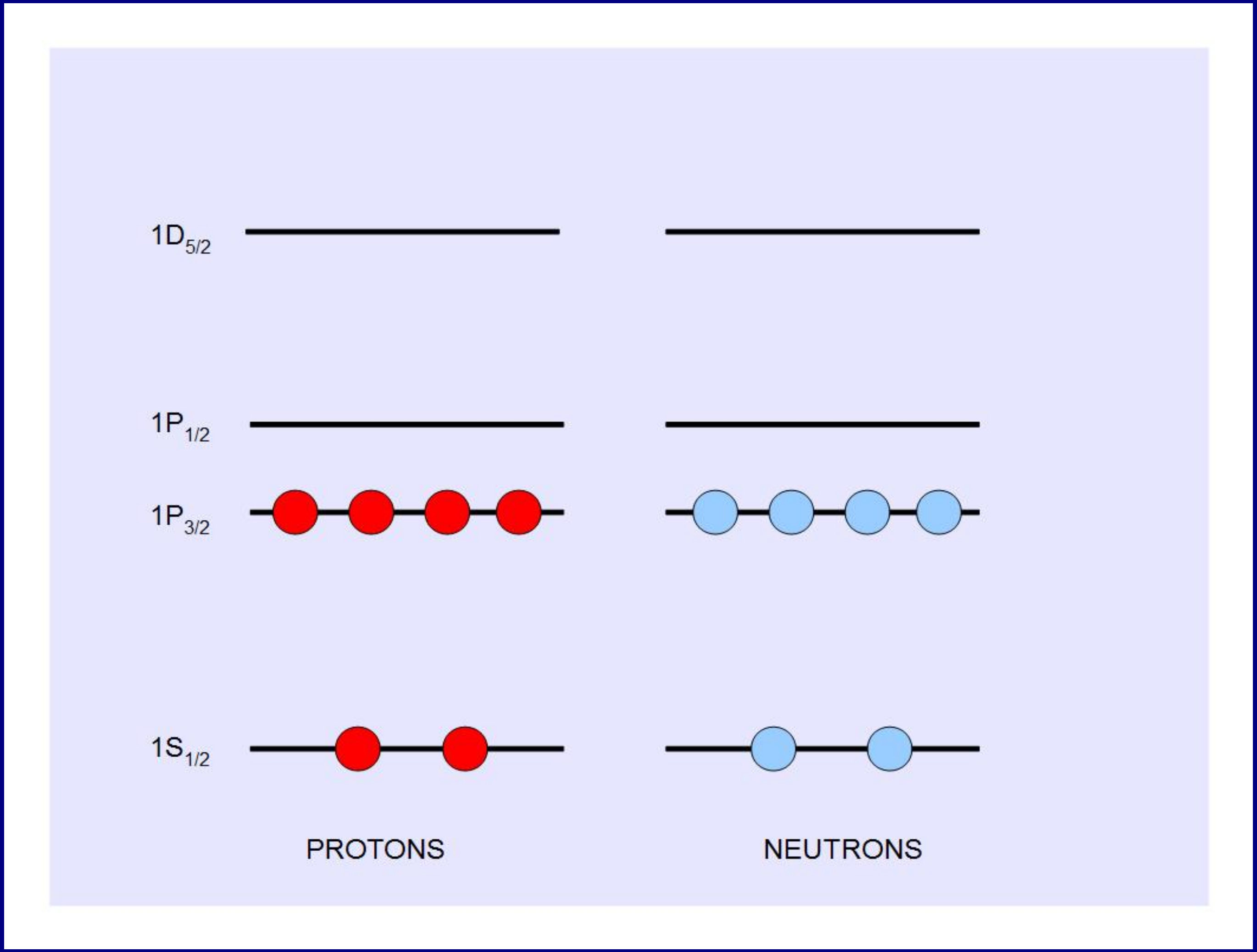

filled 1P shell

filled 1S shell

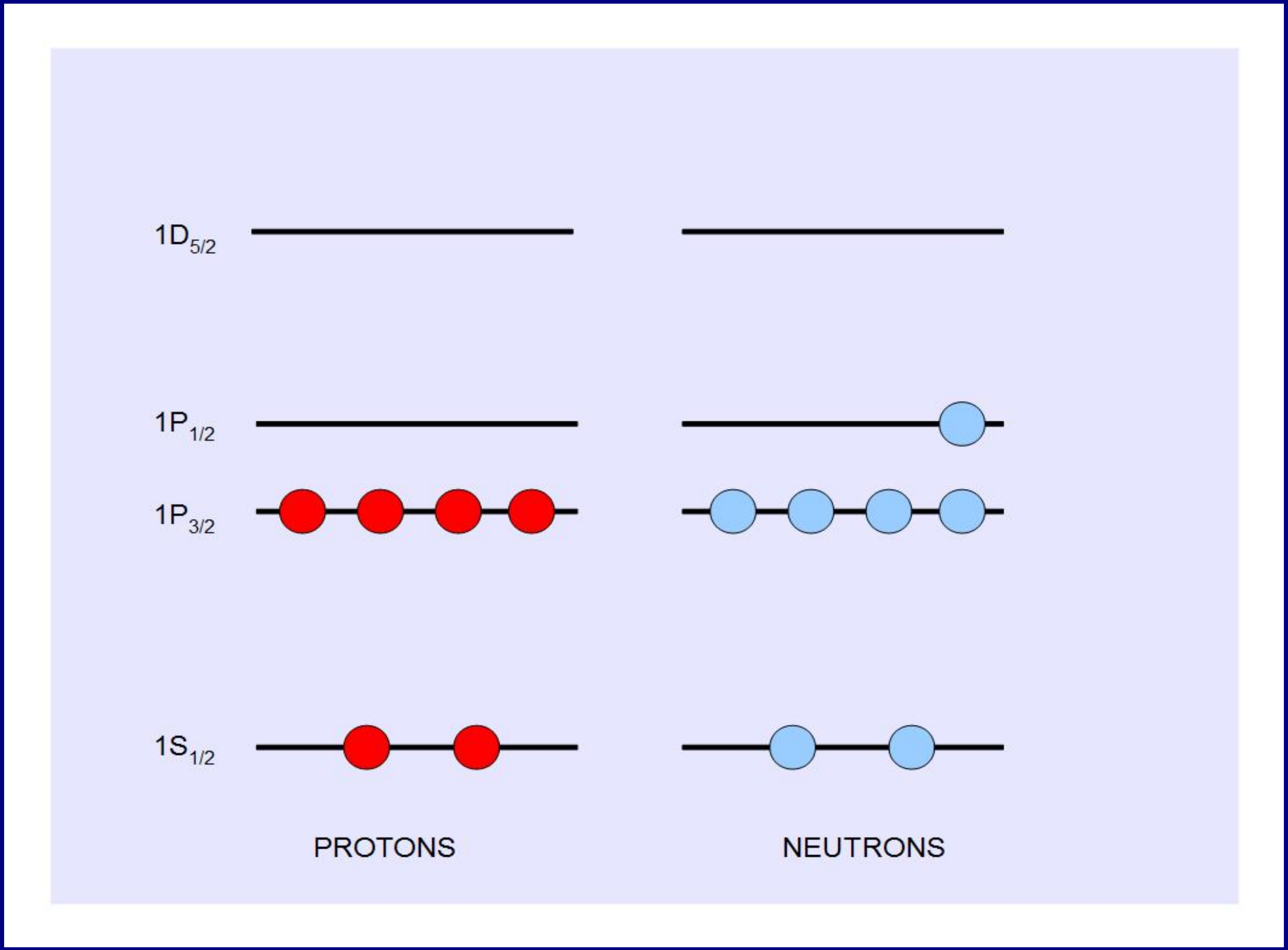
Oxygen-16 is another double closed-shell nucleus.

The protons and neutron pair off to form angular momentum J=0 pairs. Therefore, every even-even nucleus (even number of protons, even number of neutrons) has a ground state with $J^{\pi} = 0^{+}$

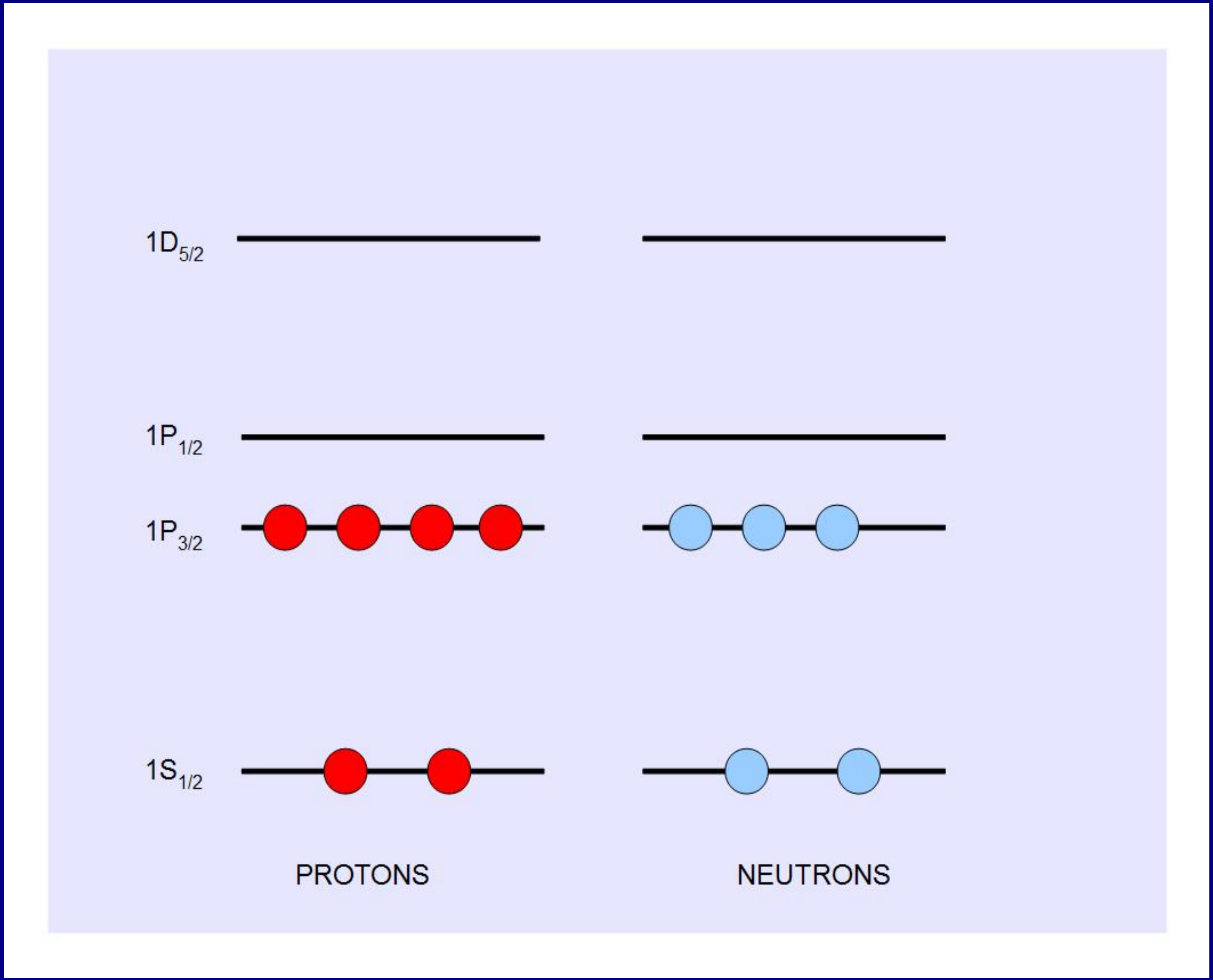
e.g. $^{12}C$ N=6 Z=6 has a $0^{+}$ ground state



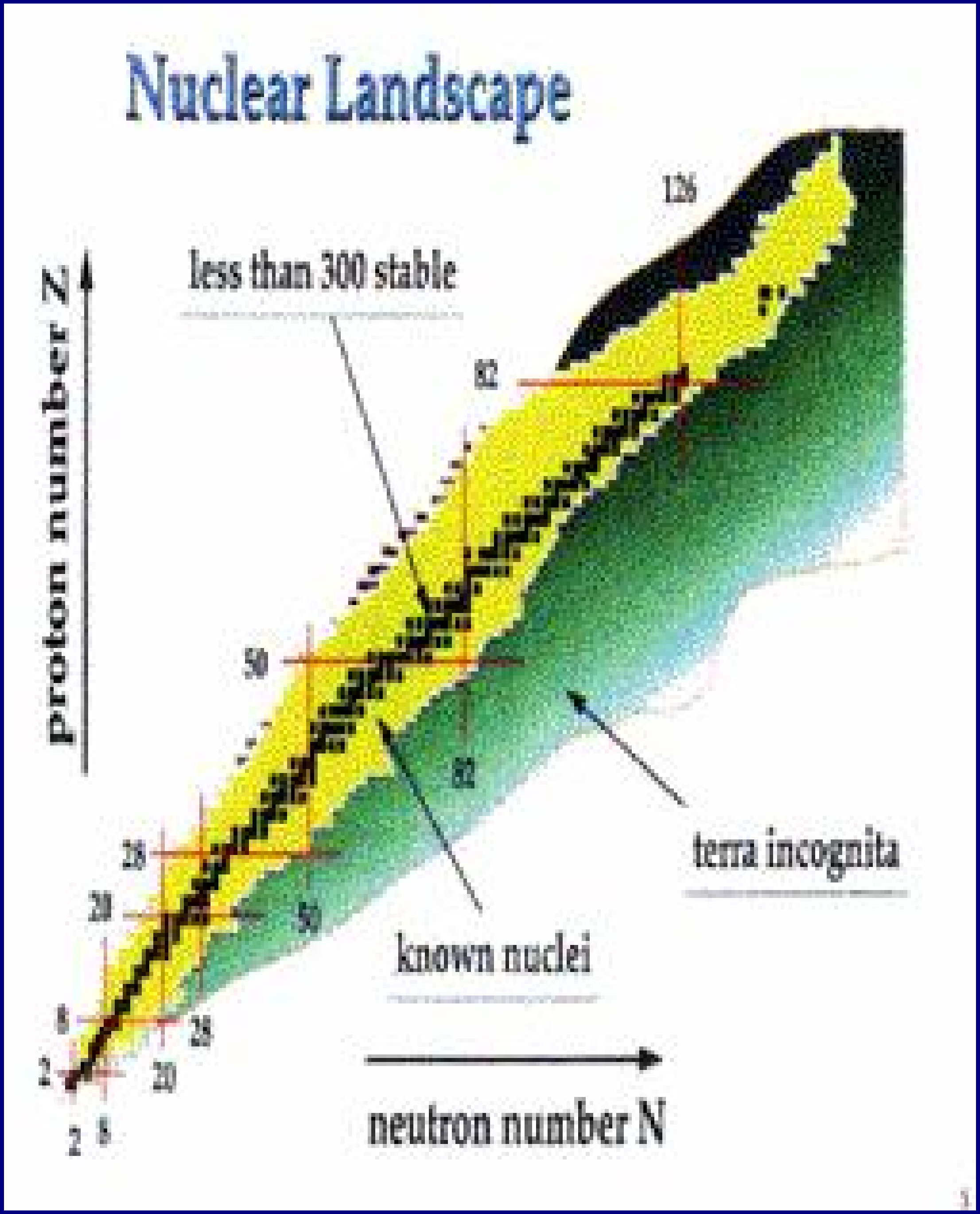
If there is a single unpaired nucleon (either proton or neutron) then the angular momentum of that single nucleon is the angular momentum of the ground state of that nucleus




Carbon-13 has one unpaired neutron in the $1P_{1/2}$ shell, so the C-13 ground state has J=1/2.




Carbon-11 has one unpaired neutronin the $1P_{3/2}$ subshell, so C-11 ground state has J=3/2.




The “magic number” that we have seen so far were obtained for nuclei close to the valley of stability.

As we move further and further from the valley of stability by adding more neutrons, the shape of the nuclear potential changes, and so the location of the “magic numbers” marking the shell closures also changes.

This is one of the frontiers that are being explored at ISAC -- how do the magic numbers evolve as we move away from the valley of stability?

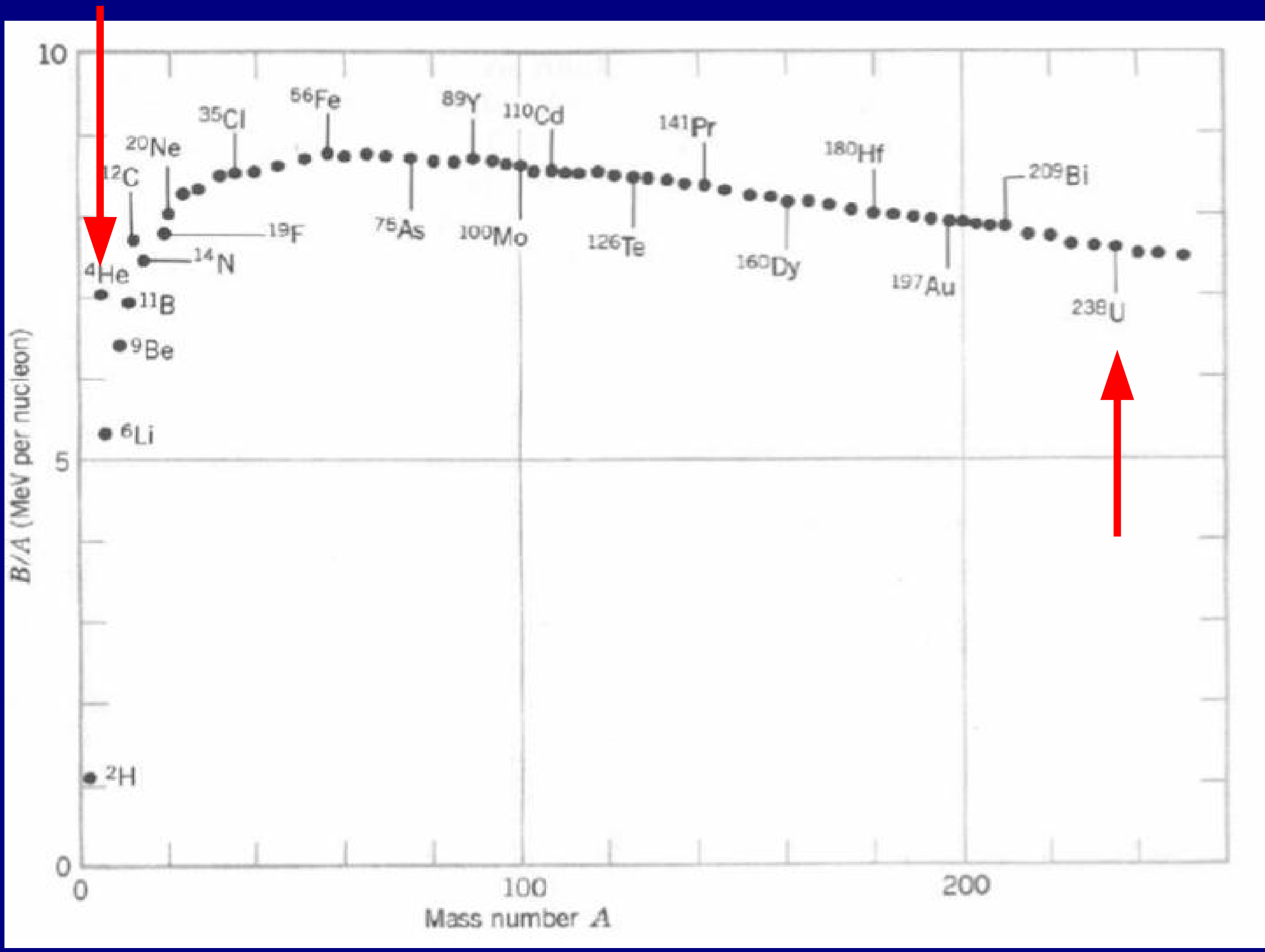
# Alpha Decay

α-particles are $^4$He nuclei

Spontaneous emission of α-particles occurs for heavy nuclei as a way for the nucleus to reduce the ratio $Z^2/A$

e.g. $^{238}\text{U} \rightarrow {}^{234}\text{Th} + \alpha$

| | | |
|---|---|---|
| Z | 92 | 90 |
| A | 238 | 234 |
| $Z^2/A$ | 35.56 | 34.61 |




Recall that it is the Coulomb repulsion ~ $Z^2$ that reduces the binding energy of heavy nuclei, while the attractive strong interaction term ~ A, so $Z^2/A$ is the ratio of repulsive Coulomb potential to attractive strong potential.

Well, if shedding charge were the aim, why doesn't the nucleus just spit out protons, or deuterons then?

The α particle is especially tightly bound (especially small mass) so that in the decay

$$^{238}U \rightarrow {}^{234}Th + \alpha$$

this gives an especially large amount of available energy to the decay product.

It is expected from “phase space” considerations,
i.e. counting the number of available
states a la statistical mechanics, that the decay rate

Rate $\sim p$ where p=momentum of the particle

and for non-relativistic particles, $E = p^2/2M$

so Rate $\sim E^{1/2}$

i.e. doubling the energy results in $\sqrt{2}$ increase in the decay rate.

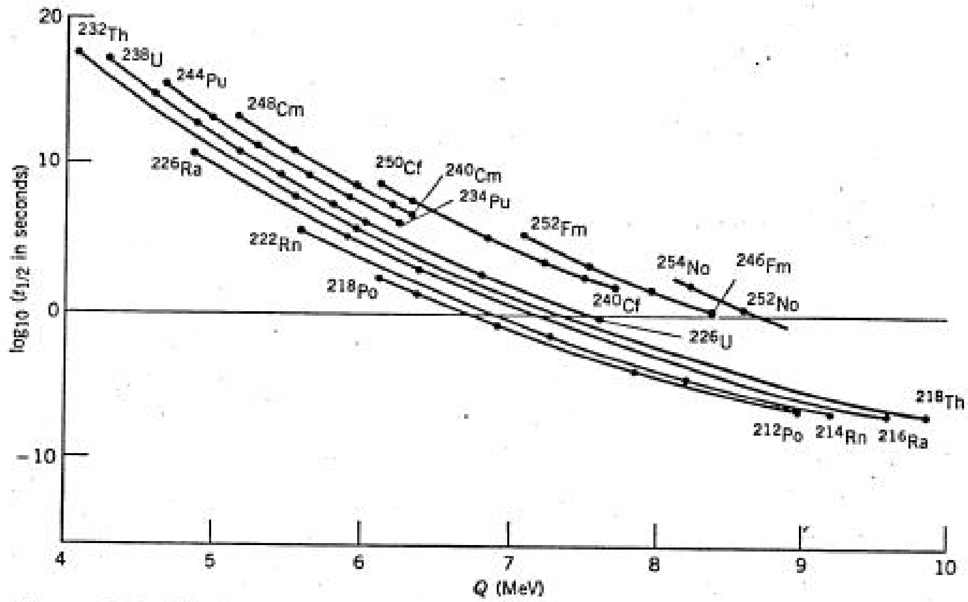



**Figure 8.1** The inverse relationship between $\alpha$-decay half-life and decay energy, called the Geiger-Nuttall rule. Only even-$Z$, even-$N$ nuclei are shown. The solid lines connect the data points.

Krane, Introductory Nuclear Physics

Geiger-Nuttall rule: large disintegration energy $\longleftrightarrow$ short half-life

Note enormous range in $t_{1/2}$

In Th isotopes, $t_{1/2}$ varies over 24 orders of magnitude

e.g. $^{232}$Th $\quad Q = 4.08$ MeV
$t_{1/2} = 1.4 \times 10^{10}$ yr

$^{218}$Th $\quad Q = 9.85$ MeV
$t_{1/2} = 1.0 \times 10^{-7}$ sec.

Q = energy released

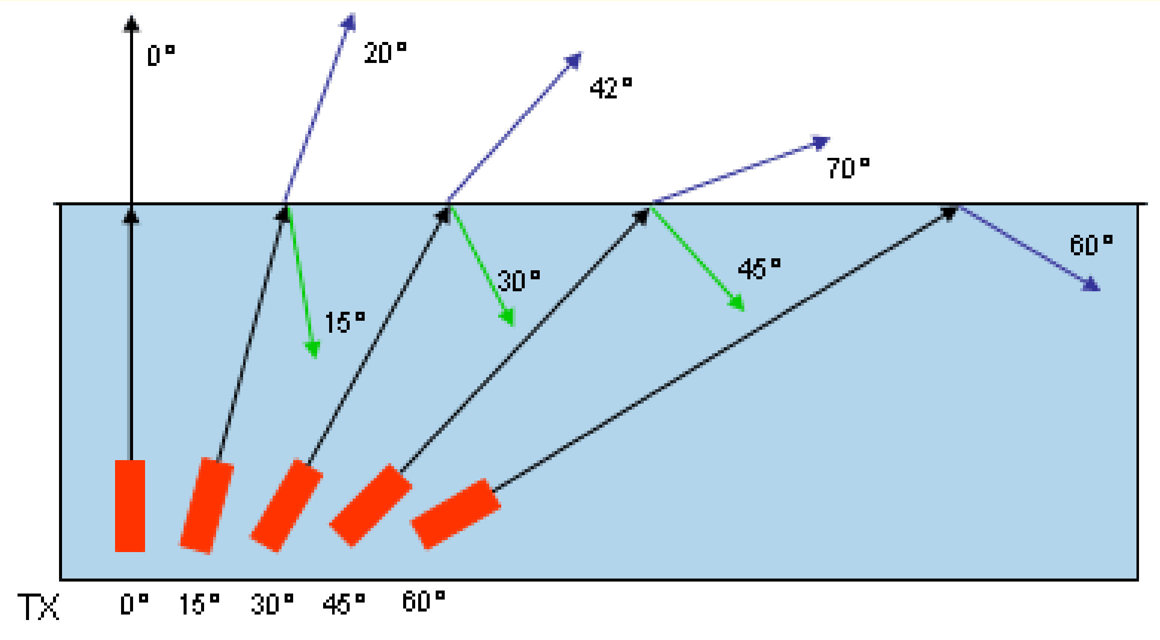
compare $^{232}$Th and $^{218}$Th

roughly double the energy, but 24 orders of magnitude change in decay rate!

This defied explanation until George Gamow realized it was the result of quantum mechanical tunneling

Quantum mechanical tunneling is just the quantum analog of evanescant waves in optics.

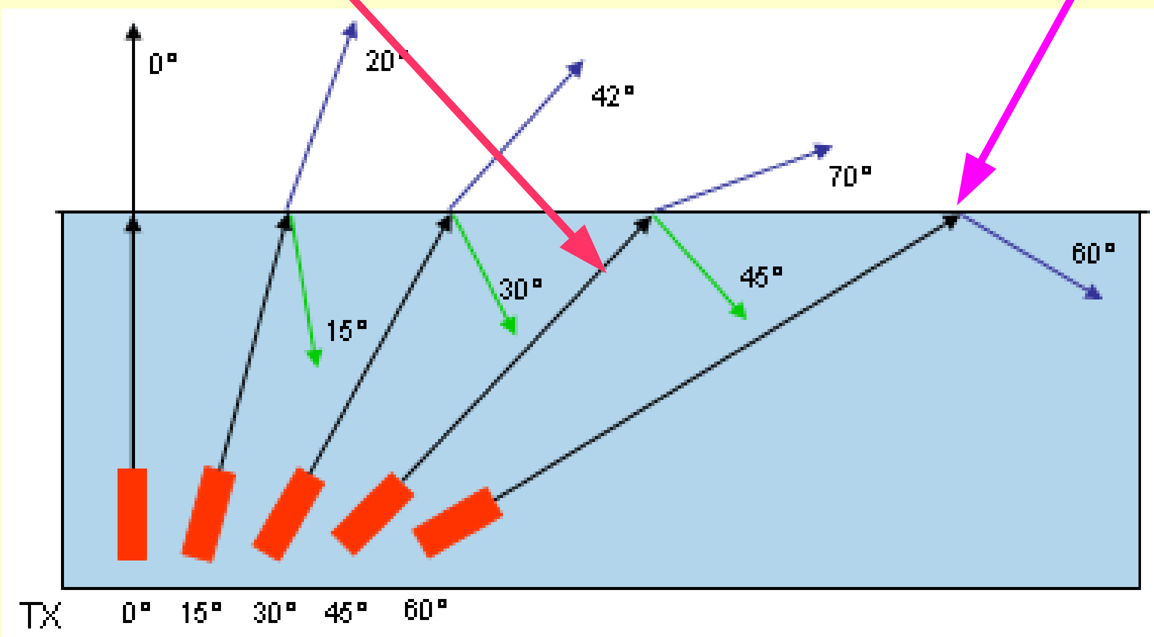
Remember Snell's law of refraction
$n_1 \sin\theta_1 = n_2 \sin\theta_2$
Total internal reflection can occur if light moves from a region of high refractive index to a region of low refractive index, when $\sin\theta_2 > 1$

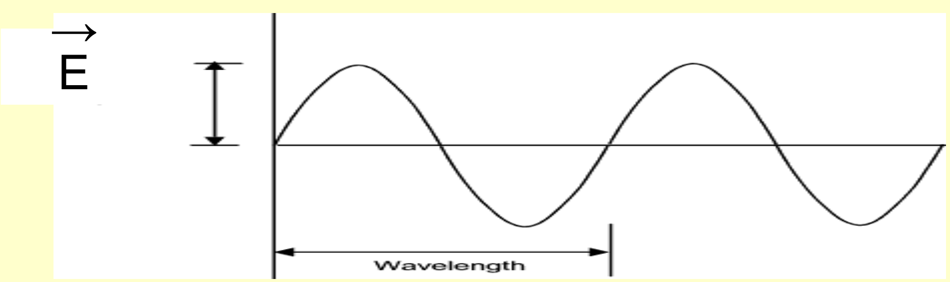


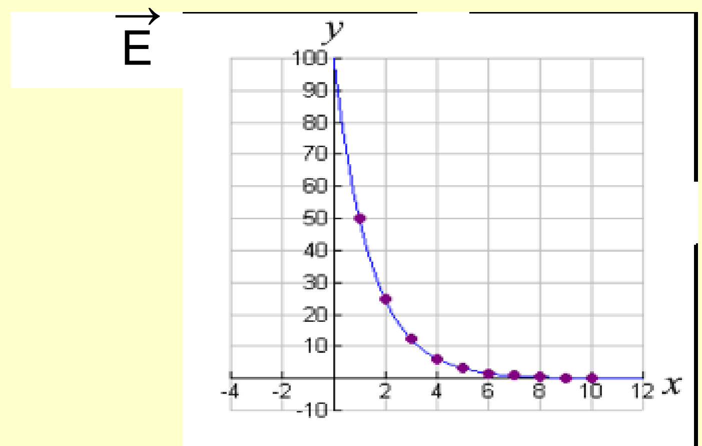
If you can actually see a real ray of light propagating freely like this, then its electric field varies like a sine wave

But what about right here? Does the electric field abruptly drop to zero at the boundary if there is total internal reflection?

The answer is NO.
The electromagnetic wave actually becomes a decaying exponential, and takes some distance to completely die away.
It looks something like this:

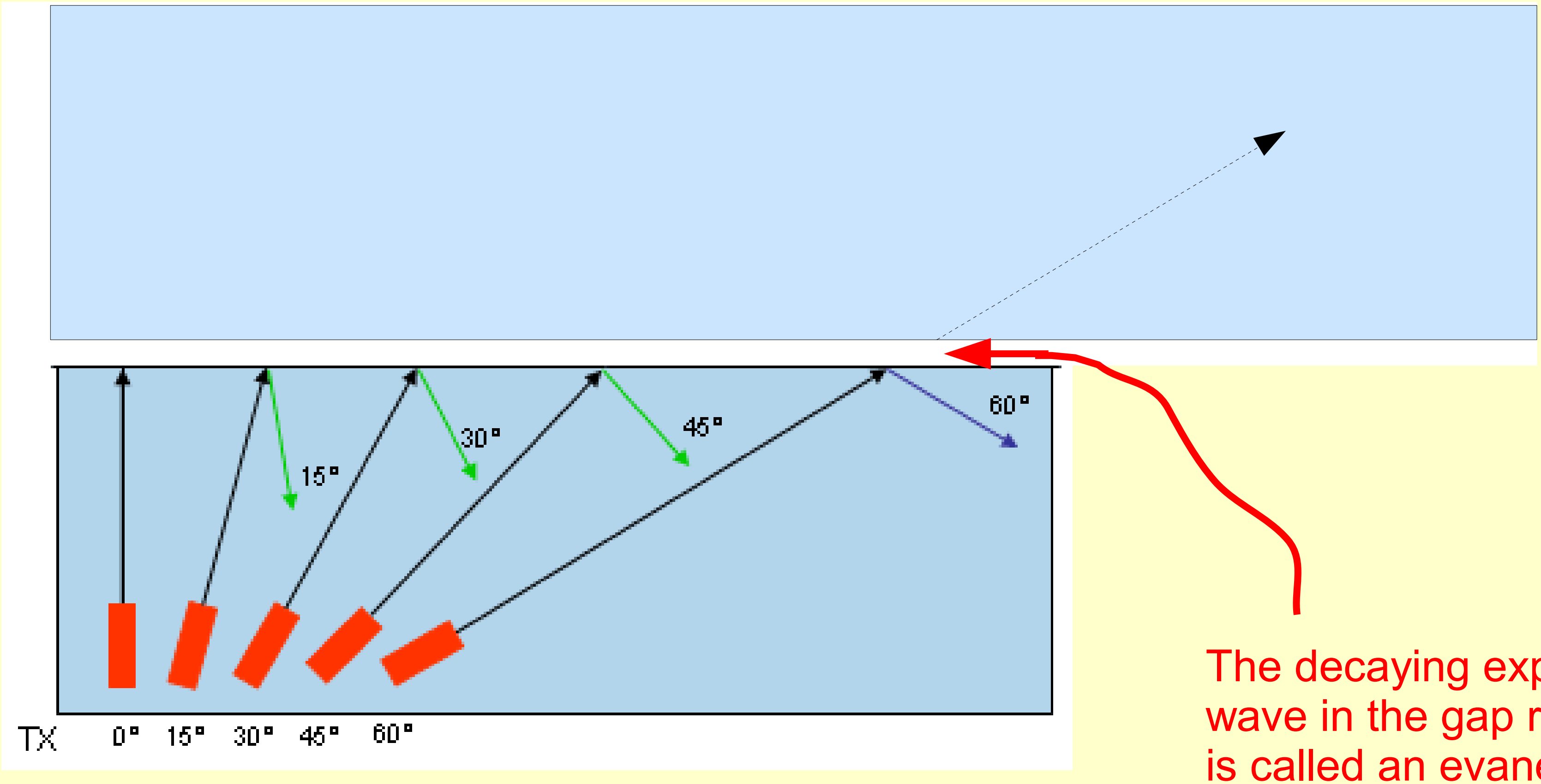



The bigger the difference in refractive index, the faster the exponential drops off.

If you hold a second block of glass very close to the first one, you will see a faint ray of light propagating into the second block, even though there is “total” internal reflection according to Snell's law. The smaller the air gap, the stronger the light in the second block, and the weaker the internal reflection. In the limit of zero air gap, all the light passes through and there is no internal reflection.

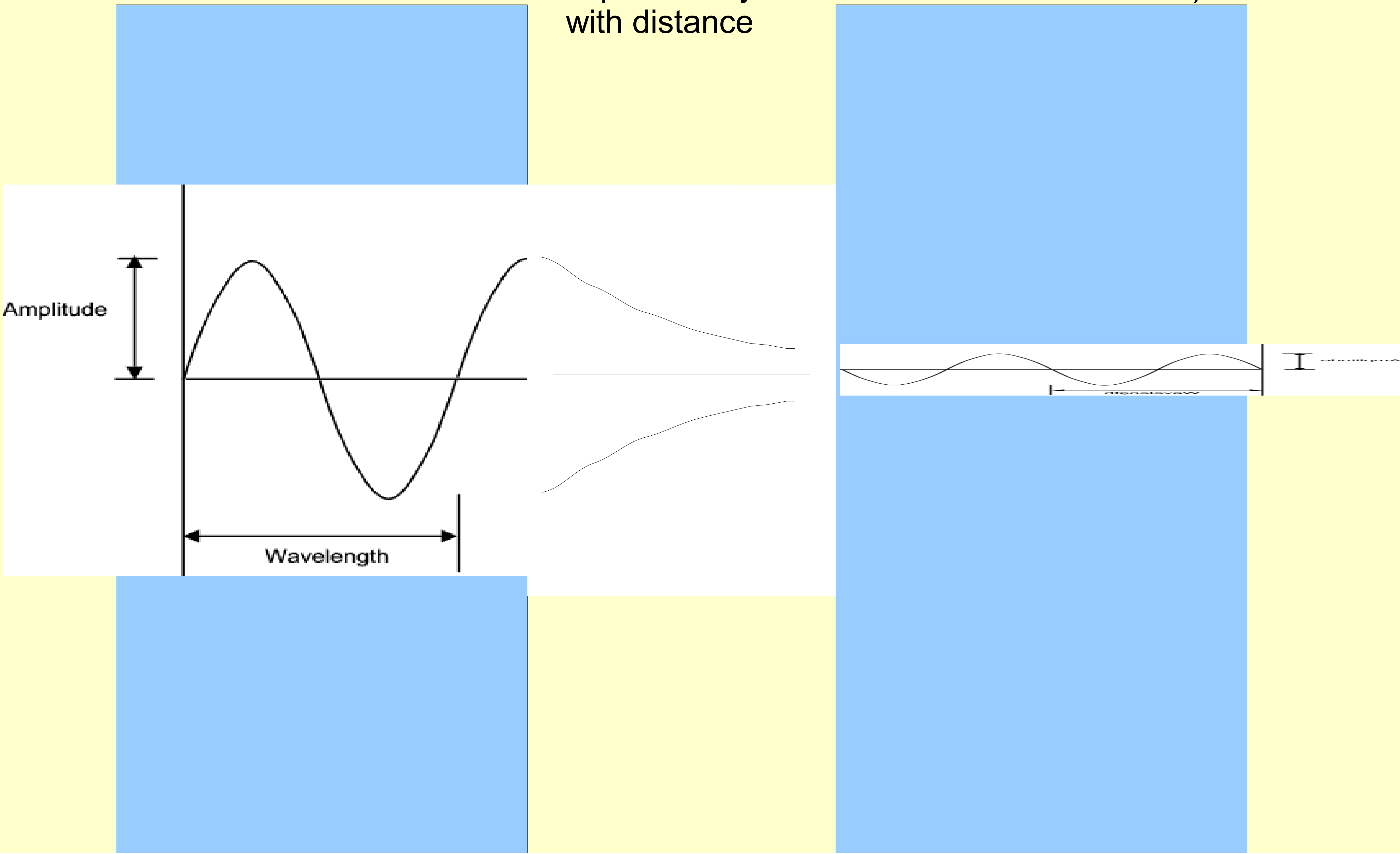



The decaying exponential wave in the gap region is called an evanescent wave.

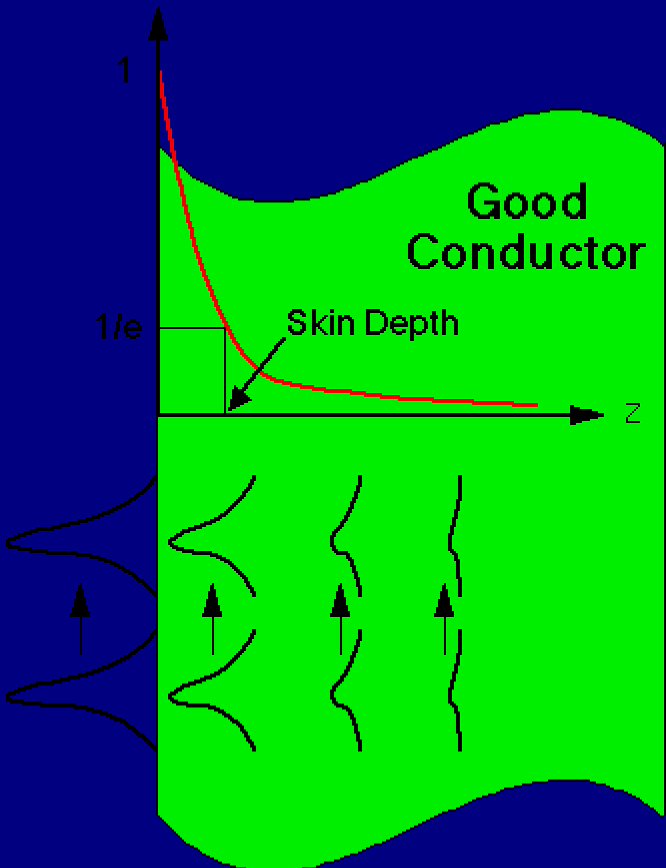
Initial glass block:
Strong light beam
(large amplitude
sinusoidal E-field)

Air gap:
Amplitude of
E-field drops
exponentially
with distance

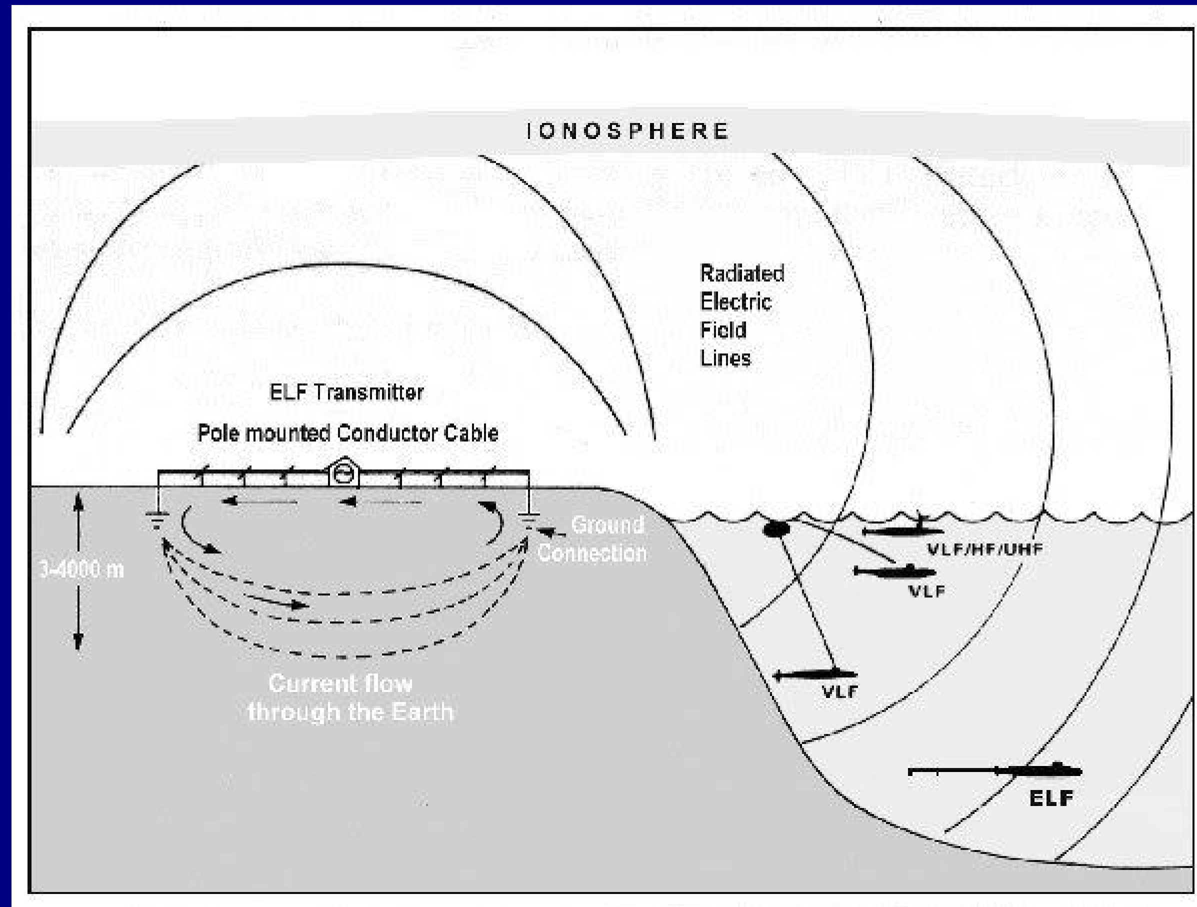
Second glass block:
Weak light beam
(small amplitude
sinusoidal E-field)



# Other examples of classical evanescent waves:

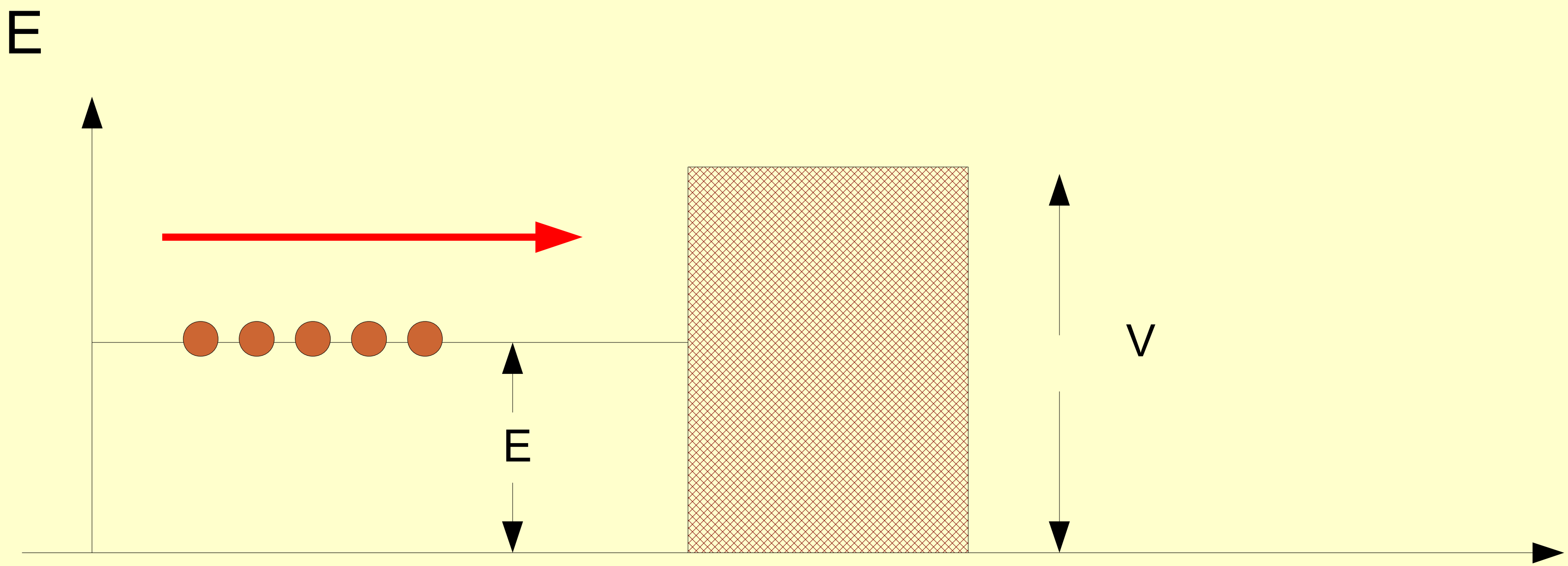



The better the conducting medium, the faster the exponential drops off.




submarines hundreds of metres beneath the sea receive communications with ELF (Extremely Low Frequency – 30-300 Hz, $\lambda=10^3$-$10^4$ km) EM waves, which penetrate into the conductive sea-water as an exponentially attenuated evanescent wave

Now consider a stream of particles
of energy E, hitting a wall (a potential energy barrier) of height V.




Classically, if E < V, then NO particles can penetrate through the wall --
they don't have enough energy to climb over the wall.

But if these are subatomic particles, the wave-like nature of the particles is important.
The particles' wavefunction does not abruptly end at the face of the wall –
the wavefunction inside the barrier (the classically forbidden region) is a
decaying exponential, just like we had in optics.
The higher or thicker the barrier, the faster the exponential falls off.

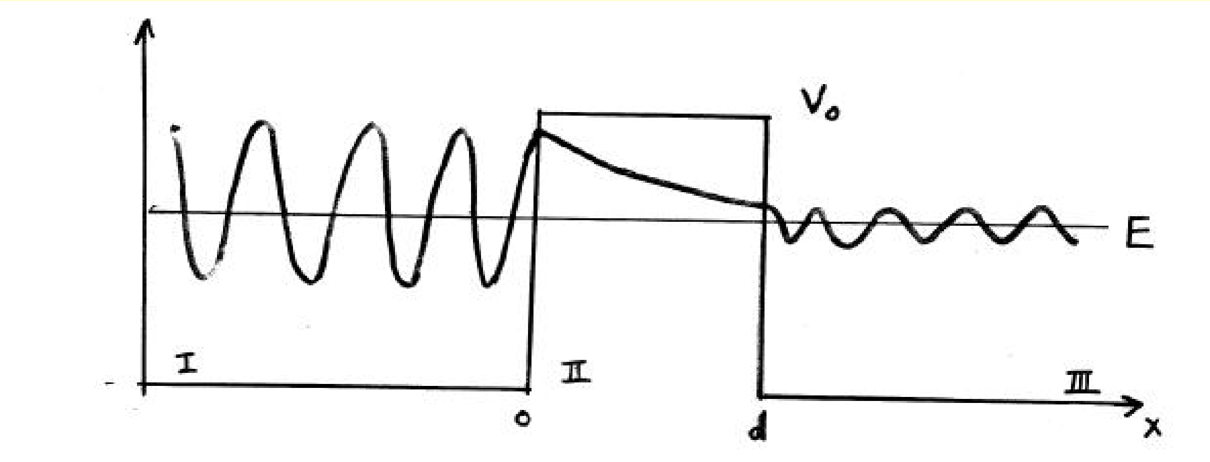
And just like the light rays in the two blocks of glass, a small fraction of the
particles will emerge on the other side of the barrier. This is quantum
mechanical tunneling!

Here's what the wavefunction looks like:

Initial: sinusoidal wave with large amplitude A

Barrier region: decaying exponential

Final: sinusoidal wave with small amplitude B




Since the probability of finding a particle is given by the square of the wavefunction's amplitude $|\psi(x)|^2$ , the initial probability is $A^2$, the final probability is $B^2$ , and the transmission probability $T = B^2 / A^2$ .

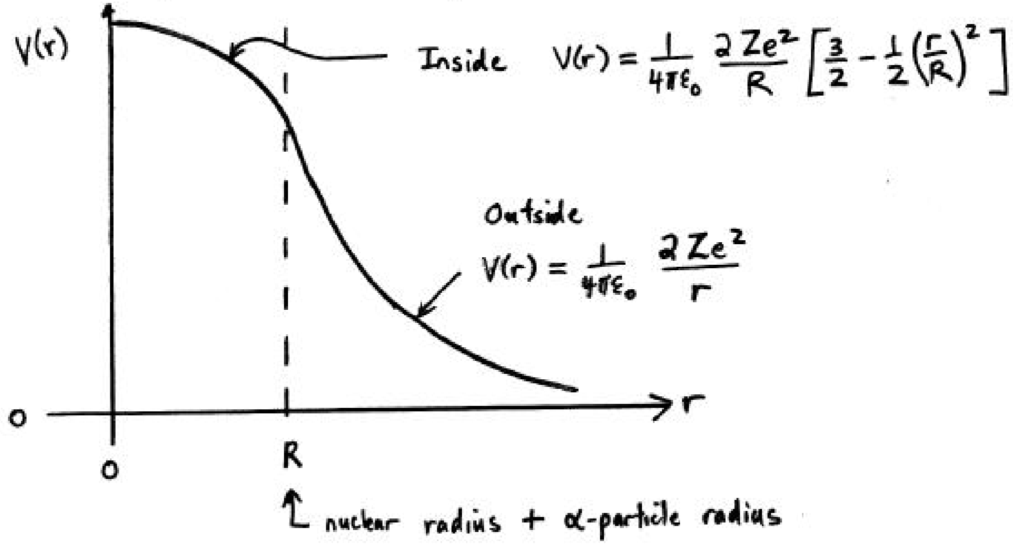
It can be shown that $T \sim \exp(-2\alpha d)$ where $\alpha = \sqrt{2m(V-E)}$ and $d$ = thickness of the barrier

i.e. the probability of penetrating the barrier drops
exponentially with the barrier thickness, and
exponentially with the square root of the height of the barrier.

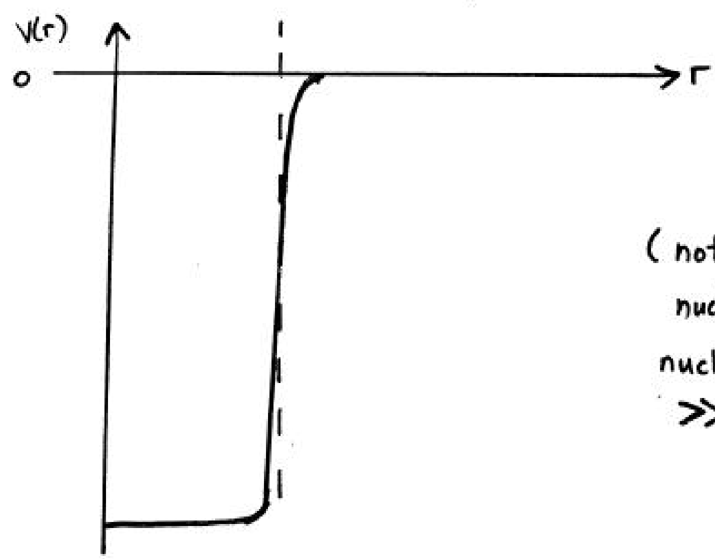
So a **small** change in energy E or thickness d makes a **huge** change in penetration.

Now consider the potential that an $\alpha$-particle feels in the vicinity of the nucleus

If the nucleus is a uniformly-charged sphere of charge $+Ze$ then the $\alpha$-particle of charge $+2e$ sees a repulsive Coulomb potential:

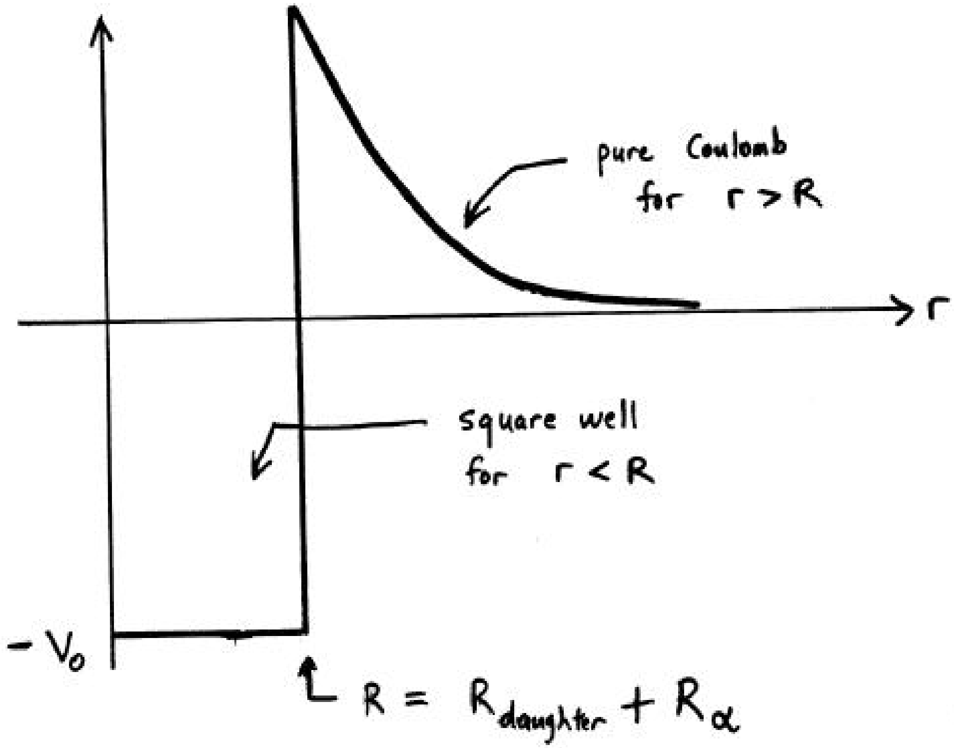



Plus an attractive nuclear potential:




(not to scale; in nuclear interior, nuclear potential $\gg$ Coulomb potential)

When we add these two together, the resulting total potential is roughly of this shape:

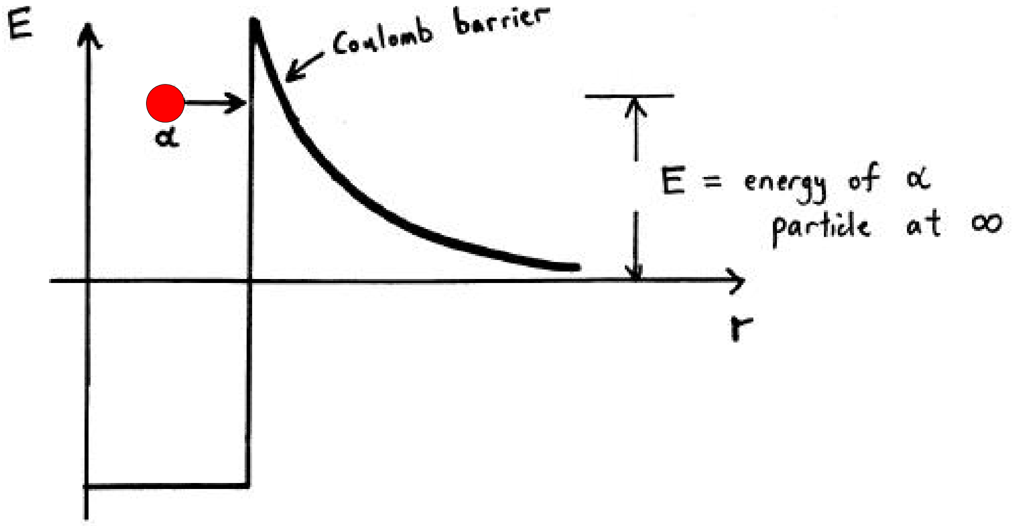



It turns out that the exact shape of the potential deep inside the nucleus doesn't matter much; it is the shape of the potential barrier that matters.

so
$$V(r) = -V_0 \quad \text{for } r < R$$

$$V(r) = \frac{1}{4\pi\varepsilon_0}\frac{2Ze^2}{r} \quad \text{for } r > R$$

Let's assume that there are pre-existing α-particles inside the nucleus, striking the Coulomb barrier from the inside. A few of them will leak out by quantum mechanical tunnelling.




Because of the hyperbolic shape of the Coulomb barrier, a higher α-particle energy E means both a <u>lower</u> barrier, and a <u>thinner</u> barrier.

Since penetration goes exponentially as the thickness, and exponentially as the square root of the barrier height, a small increase in energy E means a MUCH, MUCH larger probability that the α-particle will leak out.

For a hyperbolic-shaped barrier, it can be shown that

Transmission prob $T = \exp(-2G)$

$$= \exp(-2Z\,E^{-1/2})$$

The half-life must vary inversely with the transmission prob
(if the α particles leak out half as fast, the nuclei will live twice as long)

Half-life $t_{1/2} \sim 1/T = \exp(+2Z\,E^{-1/2})$

$\ln(t_{1/2}) \sim 2\,Z\,E^{-1/2}$

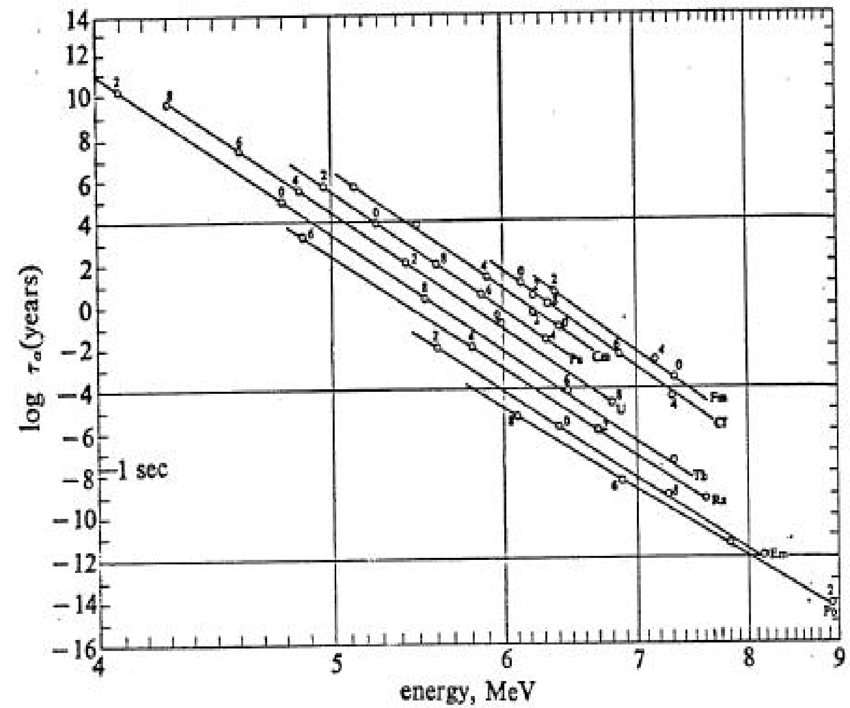



Figure 7-4 Plot of logarithms of partial alpha half-lives for ground-state transitions vs. the inverse square root of the total alpha-decay energy (alpha-particle energy + recoil energy). The points are experimental, and the straight lines are based on a least-squares analysis of the points where energies have been determined by magnetic spectrographs. The prints are numbered with the last figure of the mass number. [C. J. Gallagher and J. O. Rasmusen, *J. Inorg. Nucl. Chem.*, **3**, 333 (1957).]

from Segrè, Nuclei and Particles

Confirms linear relationship between

$\ln(t_{1/2})$ and $E^{-1/2}$

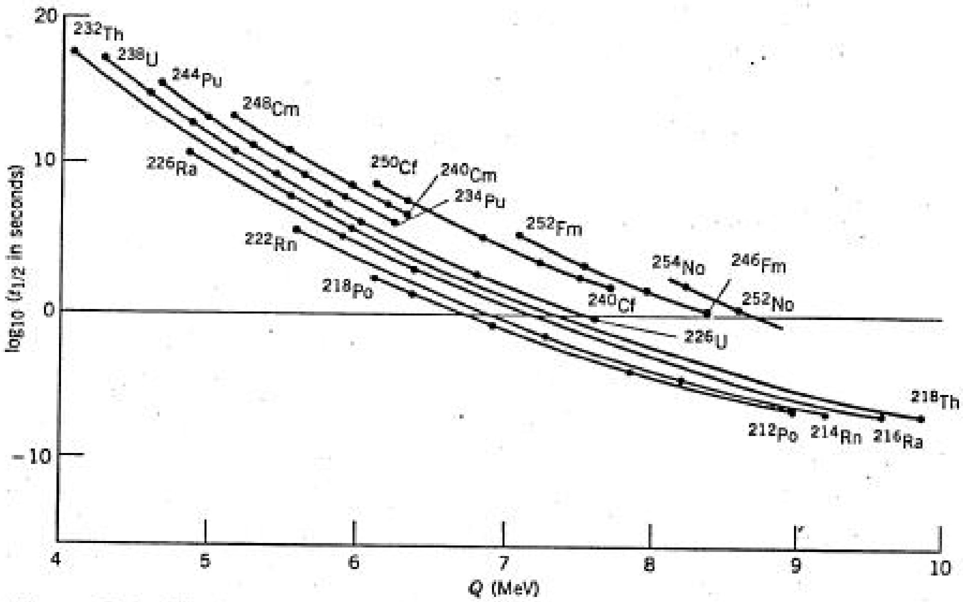



**Figure 8.1** The inverse relationship between α-decay half-life and decay energy, called the Geiger-Nuttall rule. Only even-Z, even-N nuclei are shown. The solid lines connect the data points.

Krane, Introductory Nuclear Physics

Geiger-Nuttall rule: large disintegration energy ⟷ short half-life

Note enormous range in $t_{1/2}$

In Th isotopes, $t_{1/2}$ varies over 24 orders of magnitude

e.g. $^{232}$Th $Q = 4.08$ MeV
$t_{1/2} = 1.4 \times 10^{10}$ yr

$^{218}$Th $Q = 9.85$ MeV
$t_{1/2} = 1.0 \times 10^{-7}$ sec.

Now we understand why the half-life of α-decay decreases 24 orders of magnitude when the energy just changes by a factor of 2. It's because the α particles have to quantum mechanically tunnel through the hyperbolic-shaped Coulomb barrier.

The first example of quantum tunneling that was discovered, and an inevitable consequence of wave-particle duality.

Putting in some typical numbers:

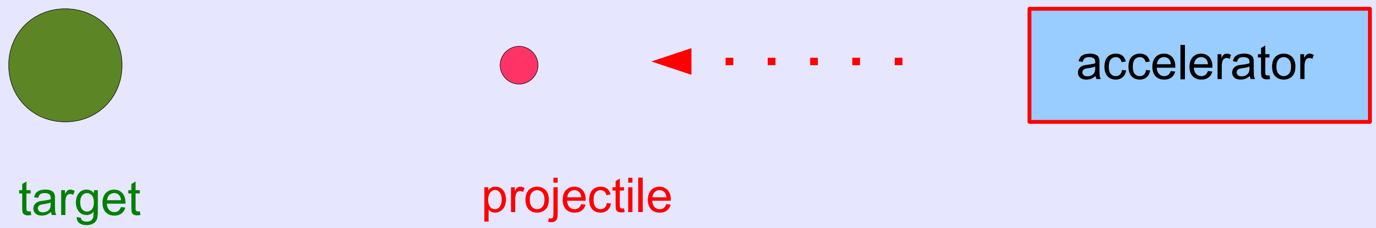
For α-particles of ~4 MeV velocity v ~ 0.46c ~ $1.4 \times 10^{7}$ m/sec
rattling around inside a thorium nucleus (diameter=14.7 fm = $1.5 \times 10^{-14}$ m)

those α-particles would traverse the diameter of the nucleus and
strike the walls ~$10^{21}$ times per second

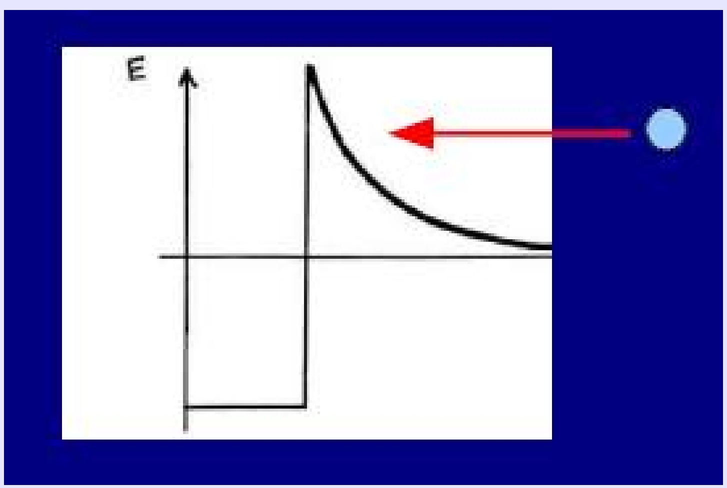
and yet it takes ~$1.4 \times 10^{10}$ years to leak out.

It is evident that the tunneling probability is very low!

Next week: Nuclear reactions




The same process for alpha particles tunneling OUT of a nucleus
also works in reverse for charged particles trying to get INTO the nucleus.




The projectiles have to quantum mechanically tunnel through the
Coulomb barrier $\rightarrow$ reaction probability increases exponentially
with the √ energy of the projectile $\rightarrow$ important astrophysical implications.


Alvin the α-particle is the epitome of the saying,
"If at first you don't succeed, try again".
He rams his head against the wall $10^{21}$ times per second
for up to $10^{10}$ years before escaping from his prison inside the nucleus.

I hope you find learning nuclear physics easier than that!